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The algebraic structure of relative twisted vertex operators

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Abstract.

Twisted vertex operators based on rational lattices have had many applications in vertex operator algebra theory and conformal field theory. In this paper, "relativized" twisted vertex operators are constructed in a general context based on isometries of rational lattices, and a generalized twisted Jacobi identity is established for them. This result generalizes many previous results. Relatived untwisted vertex operators had been studied in a monograph by the authors. The present paper includes as a special case the proof of the main relations among twisted vertex operators based on even lattices announced some time ago by the second author.

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1. Introduction

Vertex operators associated with the roots of a simple Lie algebra play an important role in the representation theory of affine Kac-Moody algebras (see for example [1, 11, 15, 17–19, 21–28, 30]). The algebraic structure of the untwisted vertex operators associated with an arbitrary even lattice L is one of the important motivations for introducing the notion of vertex (operator) algebra [2] and also provides basic examples of such algebras (see also [14]). Objects that we called "relative untwisted vertex operators" were introduced and studied in [6, 7], where the untwisted Z-algebras [21–23], which were used to construct bases of standard modules for affine Lie algebras (see also [25–28]), were placed into a systematic axiomatic context, along with parafermion algebras [32]. Roughly speaking the relative untwisted vertex operators were defined so as to "relativize" the usual untwisted vertex operators with respect to a suitable subspace of the complex span of L. The introduction of relative untwisted vertex operators led in [7] to three levels of generalization of the concept of vertex operator algebra (and module). One of these notions – that of "generalized

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vertex operator algebra" – enabled us to clarify the essential equivalence between the Z-algebras of [21,22] and the parafermion algebras of [32], among other things providing a precise mathematical foundation for "parafermion conformal field theory".

General *twisted* vertex operators associated with an arbitrary even lattice L equipped with a finite order isometry were introduced and studied in [19,13]; see also [18] for a different approach in the special case of the elements of root lattices corresponding to the roots of a simply-laced simple Lie algebra (this approach does not generalize to arbitrary root-lattice elements, which are treated in [19]). The twisted vertex operators, parametrized by an "untwisted space" V_L and acting on a "twisted space" V_L^T, satisfy a "twisted Jacobi identity" (see [13,14,19,20]). The important and suggestive case in which the isometry is -1 plays a special role in constructing the moonshine module vertex operator algebra $V^\natural$ (see [12–14]). Twisted vertex operators are closely related to orbifold theory (see [3,4,9]).

In this paper, we present a general theory of twisted vertex operators relativized with respect to a suitable subspace of the complex span of the rational lattice L, and based on lattice automorphisms, generalizing and incorporating the construction of and the results on the ordinary twisted vertex operators announced in [19,13] (see also [20]). The main result of the present paper is a generalized twisted Jacobi identity for relative twisted vertex operators. This single result generalizes many known ones. At the same time we supply detailed proofs of the main results announced in [19,13]. These results, for the case of ordinary (not relative) twisted vertex operators, were the main motivation for introducing the notion of twisted modules for a vertex algebra, in [10,5]. In particular, the twisted spaces V_L^T constructed in [19,12–14] are twisted modules for the vertex algebras V_L associated to even lattices L. Recently, our result on (ordinary) twisted vertex operators (for non-even lattices) has been used in [8] to give another construction of the moonshine module associated with a certain isometry (of order 3) of the Leech lattice. Also, twisted vertex operators based on an integral lattice have been studied in [31].

The special case in which L is the weight lattice of $sl(2,\mathbb{C})$ and the lattice isometry is -1 has been studied previously by Husu [16], where the generalized twisted Jacobi identity was established for relative twisted vertex operators. As an application of this identity, many results concerning the twisted Z-algebras discovered and developed in [25–27] in the course of constructing bases of the standard modules for the affine Lie algebra $A_1^{(1)}$ were reinterpreted in a natural way from the vertex-operator-theoretic viewpoint. As pointed out in [16], the constructions in [25–27,16] correspond to the (twisted) parafermion algebras of [33].

This paper is organized as follows: In Section 2 we present a basic setting, involving a rational lattice L and the vector space which it generates, and an isometry of finite order of the lattice. We also consider a subspace of the vector space with respect to which the vertex operators based on the untwisted space V_L will be "relativized". In Section 3, we recall the notion of relative untwisted vertex operator (see [7]). These operators are parametrized by the space V_L and also act on this space. Relative twisted vertex operators, parametrized by the untwisted space and acting on the twisted space

V_L^T, are defined in Section 4. A fundamental quadratic operator Δ_z enters into the construction of these operators, as in [13, 14].

Section 5, the central part of this work, is devoted to the formulation and proof of our generalized twisted Jacobi identity for relative twisted vertex operators. In Section 6, we focus our attention on certain relative (untwisted and twisted) vertex operators providing representations of the Virasoro algebra, and we also compute the weights of the twisted spaces. The quadratic operator Δ_z plays a basic role here. We observe that the resulting shift in conformal weight associated with the twisted space is related to certain values of the second Bernoulli polynomial. Finally, in Section 7, we put the twisted spaces V_L^T constructed in [19, 13] (see also [20]) into the axiomatic context of twisted modules for the vertex algebra V_L.

2. The setting

In this section we introduce notation and assumptions. In particular, we introduce a lattice L with an isometry ν, two central extensions $\hat{L}$ and $\hat{L}_\nu$ by the same finite cyclic group, and a subspace $\boldsymbol{h}_*$ of the complex span $\boldsymbol{h}$ of L. The lattice L together with the central extension $\hat{L}$ will be used to build a vector space V_L which will parametrize relative untwisted vertex operators and on which these operators will also act. The space V_L will also parametrize relative twisted vertex operators acting on a vector space V_L^T that will be built from L and the central extension $\hat{L}_\nu$.

We work in the setting of [19, 13, 7]. Let p and q be two fixed positive integers such that p divides q. The following data and conditions are assumed:

2.1 Let L be a (rational) lattice equipped with a symmetric nondegenerate $\mathbb{Q}$-valued $\mathbb{Z}$-bilinear form $\langle \cdot, \cdot \rangle$, not necessarily positive definite.

2.2. Let ν be an isometry of L such that $\nu^p = 1$ (but p need not be the exact order of ν and we may take $\nu = 1$).

2.3. Let c_0 and c_0^ν be two ν-invariant alternating $\mathbb{Z}$-bilinear maps

$$L \times L \to \mathbb{Z}/q\mathbb{Z}. \tag{2.1}$$

Remark 2.1. For a positive integer n let $\langle \kappa_n \rangle$ be the cyclic group of order n with generator κ_n. We assume that the generators are related by the condition $\kappa_n = \kappa_m^{m/n}$ if n divides m. (This holds if for instance $\kappa_n = e^{2\pi i/n}$ for all n.) Viewed as alternating $\mathbb{Z}$-bilinear maps from $L \times L$ to $\mathbb{Z}/q\mathbb{Z}$, c_0 and c_0^ν determine two central extensions of L,

$$1 \to \langle \kappa_q \rangle \to \hat{L} \xrightarrow{\;-\;} L \to 1, \qquad 1 \to \langle \kappa_q \rangle \to \hat{L}_\nu \xrightarrow{\;-\;} L \to 1, \tag{2.2}$$

uniquely up to equivalence, by the commutator conditions

$$aba^{-1}b^{-1} = \kappa_q^{c_0(\bar{a}, \bar{b})}, \qquad aba^{-1}b^{-1} = \kappa_q^{c_0^\nu(\bar{a}, \bar{b})} \in \langle \kappa_q \rangle \tag{2.3}$$

for $a, b \in \hat{L}$ or $a, b \in \hat{L}_\nu$ (cf. for example [14, Section 5.2]). Then there is a set-theoretic identification (which is usually not an isomorphism of groups) between the groups $\hat{L}$ and $\hat{L}_\nu$ such that the respective group multiplications $\times$ and $\times_\nu$ are related by

$$a \times b = \kappa_q^{\varepsilon_0(\bar{a}, \bar{b})} a \times_\nu b, \tag{2.4}$$

where ε_0 is a bilinear map from $L \times L$ to $\mathbb{Z}/q\mathbb{Z}$ satisfying

$$\varepsilon_0(\alpha, \beta) - \varepsilon_0(\beta, \alpha) = c_0(\alpha, \beta) - c_0^\nu(\alpha, \beta) \tag{2.5}$$

for $\alpha, \beta \in L$. (To see this, we may choose a section of $\hat{L}$ and a section of $\hat{L}_\nu$ such that the corresponding 2-cocycles are bilinear and we then take ε_0 to be the difference of the two cocycles. The identification is evident.) Moreover, ν lifts to an automorphism $\hat{\nu}$ of $\hat{L}$ fixing κ_q, that is,

$$\overline{\hat{\nu} a} = \nu \bar{a} \quad \text{for } a \in \hat{L}, \qquad \hat{\nu}\kappa_q = \kappa_q, \tag{2.6}$$

and such a lifting is unique up to multiplication by a lifting of the identity automorphism of L, which is of the form

$$\begin{aligned} \lambda^* &: \hat{L} \to \hat{L} \\ &a \mapsto a\kappa_q^{\lambda(\bar{a})} \end{aligned} \tag{2.7}$$

for some $\lambda \in \mathrm{Hom}(L, \mathbb{Z}/q\mathbb{Z})$ (cf. [14], Section 5.4]). If $\nu = 1$, then we may take $\hat{\nu} = 1$. Note that $\hat{\nu}$ also acts set-theoretically on $\hat{L}_\nu$ by using the identification. Moreover, if $\varepsilon_0(\cdot, \cdot)$ is ν-invariant, then $\hat{\nu}$ is also an automorphism of $\hat{L}_\nu$, from (2.4).

Remark 2.2. Here we describe the special case which is the subject of [19]; see also Sections 3.1 and 3.3 of [13]. Let L be an even lattice, i.e., $\langle \alpha, \alpha \rangle \in 2\mathbb{Z}$ for all $\alpha \in L$. Take $q = p$ if p is even and take $q = 2p$ if p is odd, so that q is always even. We assume that

$$\langle \nu^{p/2}\alpha, \alpha \rangle \in 2\mathbb{Z} \quad \text{for } \alpha \in L \tag{2.8}$$

if p is even; this can always be arranged by doubling p if necessary. Then

$$c_0(\alpha, \beta) = q\langle \alpha, \beta \rangle/2 + q\mathbb{Z}, \tag{2.9}$$

$$c_0^\nu(\alpha, \beta) = \sum_{r=0}^{p-1} (q/2 + qr/p)\langle \nu^r\alpha, \beta \rangle + q\mathbb{Z} \tag{2.10}$$

for $\alpha, \beta \in L$ define two ν-invariant alternating $\mathbb{Z}$-bilinear maps from $L \times L$ to $\mathbb{Z}/q\mathbb{Z}$. We have the commutator relations

$$aba^{-1}b^{-1} = \kappa_2^{\langle \bar{a}, \bar{b} \rangle} \tag{2.11}$$

in $\hat{L}$ and

$$aba^{-1}b^{-1} = \kappa_2^{\sum_{r=0}^{p-1}\langle \nu^r\bar{a}, \bar{b} \rangle} \kappa_p^{\sum_{r=0}^{p-1} r\langle \nu^r\bar{a}, \bar{b} \rangle} \tag{2.12}$$

in $\hat{L}_\nu$. We may take

$$\varepsilon_0(\alpha,\beta) = \sum_{0<r<p/2} (q/2 + qr/p)\langle \nu^{-r}\alpha,\beta\rangle + q\mathbb{Z} \tag{2.13}$$

in this case. Note that $\varepsilon_0(\cdot,\cdot)$ is ν-invariant and thus $\hat{\nu}$ is also an automorphism of $\hat{L}_\nu$. It is shown in [19] that there is a lifting $\hat{\nu}$ of ν with the special property that for $a \in \hat{L}$, $\hat{\nu}a = a$ if $\nu\bar{a} = \bar{a}$. In the special cases $p = 1$ and $p = 2$, we have $c_0^\nu = c_0$, $\varepsilon_0 = 0$, and $\hat{L}_\nu = \hat{L}$ as groups.

2.4. Let $\boldsymbol{h}_*$ be a subspace (possibly 0 or $\boldsymbol{h}$) of

$$\boldsymbol{h} = \mathbb{C} \otimes_{\mathbb{Z}} L \tag{2.14}$$

on which the natural (nonsingular) extension of the form $\langle\cdot,\cdot\rangle$ on L, still denoted $\langle\cdot,\cdot\rangle$, remains nonsingular. That is,

$$\boldsymbol{h} = \boldsymbol{h}_* \oplus \boldsymbol{h}_*^\perp, \tag{2.15}$$

$^\perp$ denoting orthogonal complement. We write

$$\begin{aligned} &\boldsymbol{h} \to \boldsymbol{h}_*^\perp, \qquad \boldsymbol{h} \to \boldsymbol{h}_* \\ &h \mapsto h', \qquad h \mapsto h'' \end{aligned} \tag{2.16}$$

for the projection maps to $\boldsymbol{h}_*^\perp$ and $\boldsymbol{h}_*$. We also assume that $\boldsymbol{h}_*$ is stable under the natural action of ν on $\boldsymbol{h}$:

$$\nu\boldsymbol{h}_* = \boldsymbol{h}_*. \tag{2.17}$$

Then the two projection maps commute with the action of ν.

3. Relative untwisted vertex operators

We shall define the untwisted space V_L (cf. [14]) and present the notion of relative untwisted vertex operators $Y_*(v,z)$ ($v \in V_L$) acting on V_L, following [6, 7]. This section, almost a copy of Ch. 3 of [7], is supplied for the reader's convenience. The degenerate case $\boldsymbol{h}_* = 0$ (the "unrelativized" case) is important in its own right.

The affine Lie algebra associated with the abelian Lie algebra $\boldsymbol{h}$ is given by

$$\hat{\boldsymbol{h}} = \boldsymbol{h} \otimes \mathbb{C}[t,t^{-1}] \otimes \mathbb{C}c \tag{3.1}$$

with structure defined by

$$[x \otimes t^m, y \otimes t^n] = \langle x,y\rangle m\delta_{m+n,0}c \quad \text{for } x,y \in \boldsymbol{h},\ m,n \in \mathbb{Z}, \tag{3.2}$$

$$[c,\hat{\boldsymbol{h}}] = 0. \tag{3.3}$$

(This notation and related notation below may be applied to any finite-dimensional abelian Lie algebra with a nonsingular symmetric form.) By (2.15) we have

$$\hat{\boldsymbol{h}} = \boldsymbol{h}_* \otimes \mathbb{C}[t,t^{-1}] \oplus \boldsymbol{h}_*^\perp \otimes \mathbb{C}[t,t^{-1}] \oplus \mathbb{C}c. \tag{3.4}$$

Then $\hat{\boldsymbol{h}}$ has a $\mathbb{Z}$-gradation, the *weight gradation* associated with $\boldsymbol{h}_*$, given by

$$\mathrm{wt}(x \otimes t^m) = 0, \qquad \mathrm{wt}(y \otimes t^n) = -n, \qquad \mathrm{wt}\, c = 0 \tag{3.5}$$

for $x \in \boldsymbol{h}_*, y \in \boldsymbol{h}_*^{\perp}$ and $m, n \in \mathbb{Z}$.

Set

$$\hat{\boldsymbol{h}}^+ = \boldsymbol{h} \otimes t\mathbb{C}[t], \qquad \hat{\boldsymbol{h}}^- = \boldsymbol{h} \otimes t^{-1}\mathbb{C}[t^{-1}]. \tag{3.6}$$

The subalgebra

$$\hat{\boldsymbol{h}}_{\mathbb{Z}} = \hat{\boldsymbol{h}}^+ \oplus \hat{\boldsymbol{h}}^- \oplus \mathbb{C}c \tag{3.7}$$

of $\hat{\boldsymbol{h}}$ is a Heisenberg algebra. Consider the induced $\hat{\boldsymbol{h}}$-module, irreducible even under $\hat{\boldsymbol{h}}_{\mathbb{Z}}$,

$$M(1) = U(\hat{\boldsymbol{h}}) \otimes_{U(\boldsymbol{h} \otimes \mathbb{C}[t] \oplus \mathbb{C}c)} \mathbb{C} \simeq S(\hat{\boldsymbol{h}}^-) \quad \text{(linearly)}, \tag{3.8}$$

$\boldsymbol{h} \otimes \mathbb{C}[t]$ acting trivially on $\mathbb{C}$ and c acting as 1; $U(\cdot)$ denotes universal enveloping algebra and $S(\cdot)$ denotes symmetric algebra. The $\hat{\boldsymbol{h}}$-module $M(1)$ is $\mathbb{Z}$-graded so that $\mathrm{wt}\, 1 = 0$ (we write 1 for $1 \otimes 1$):

$$M(1) = \coprod_{n \in \mathbb{Z}, n \geq 0} M(1)_n, \tag{3.9}$$

where $M(1)_n$ denotes the homogeneous subspace of weight n. The automorphism ν of L acts in a natural way on $\boldsymbol{h}$, on $\hat{\boldsymbol{h}}$ (fixing c) and on $M(1)$, preserving the gradations, and for $u \in \hat{\boldsymbol{h}}$ and $m \in M(1)$,

$$\nu(u \cdot m) = \nu(u) \cdot \nu(m). \tag{3.10}$$

Form the induced $\hat{L}$-module and $\mathbb{C}$-algebra

$$\begin{aligned} \mathbb{C}\{L\} &= \mathbb{C}[\hat{L}]/(\kappa_q - \omega_q)\mathbb{C}[\hat{L}] \\ &= \mathbb{C}[\hat{L}] \otimes_{\mathbb{C}[\langle \kappa_q \rangle]} \mathbb{C} \simeq \mathbb{C}[L] \quad \text{(linearly)}, \end{aligned} \tag{3.11}$$

where $\mathbb{C}[\cdot]$ denotes group algebra, ω_q is a primitive qth root of unity in $\mathbb{C}^\times$ and κ_q acts as ω_q on $\mathbb{C}$. For $a \in \hat{L}$, write $\iota(a)$ for the image of a in $\mathbb{C}\{L\}$. Then the action of $\hat{L}$ on $\mathbb{C}\{L\}$ and the product in $\mathbb{C}\{L\}$ are given by

$$a \cdot \iota(b) = \iota(a)\iota(b) = \iota(ab), \tag{3.12}$$

$$\kappa_q \cdot \iota(b) = \omega_q \iota(b) \tag{3.13}$$

for $a, b \in \hat{L}$. We given $\mathbb{C}\{L\}$ the $\mathbb{C}$-gradation determined by

$$\mathrm{wt}\, \iota(a) = \frac{1}{2} \langle \bar{a}', \bar{a}' \rangle \quad \text{for } a \in \hat{L}. \tag{3.14}$$

The automorphism $\hat{\nu}$ of $\hat{L}$ acts canonically on $\mathbb{C}\{L\}$, preserving the gradation, in such a way that

$$\hat{\nu}\iota(a) = \iota(\hat{\nu}a) \quad \text{for } a \in \hat{L}, \tag{3.15}$$

and we have

$$\hat{\nu}(\iota(a)\iota(b)) = \hat{\nu}(a\cdot\iota(b)) = \hat{\nu}(a)\cdot\hat{\nu}\iota(b) = \hat{\nu}\iota(a)\hat{\nu}\iota(b). \tag{3.16}$$

Also define a grading-preserving action of $\boldsymbol{h}$ on $\mathbb{C}\{L\}$ by

$$h\cdot\iota(a) = \langle h', \bar{a}\rangle\iota(a) \tag{3.17}$$

for $h \in \boldsymbol{h}$ (so that $\boldsymbol{h}_*$ acts trivially). Then $\boldsymbol{h}$ acts as algebra derivations and

$$\hat{\nu}(h\cdot\iota(a)) = \nu(h)\cdot\hat{\nu}\iota(a). \tag{3.18}$$

We shall use a formal variable z (and later, commuting formal variables z, z_0, z_1, z_2, etc.). Define

$$z^h\cdot\iota(a) = z^{\langle h', \bar{a}\rangle}\iota(a) \tag{3.19}$$

for $h \in \boldsymbol{h}$. Then

$$\hat{\nu}(z^h\cdot\iota(a)) = z^{\nu(h)}\cdot\hat{\nu}\iota(a). \tag{3.20}$$

We shall mostly be interested in the actions of h and z^h on $\mathbb{C}\{L\}$ only for $h \in \boldsymbol{h}_*^\perp$.

Set

$$V_L = M(1)\otimes_{\mathbb{C}}\mathbb{C}\{L\} \simeq S(\hat{\boldsymbol{h}}^-)\otimes\mathbb{C}[L] \quad \text{(linearly)} \tag{3.21}$$

and give V_L the tensor produce $\mathbb{C}$-gradation:

$$V_L = \coprod_{n\in\mathbb{C}} (V_L)_n. \tag{3.22}$$

We have $\mathrm{wt}\,\iota(1) = 0$, where we identify $\mathbb{C}\{L\}$ with $1\otimes\mathbb{C}\{L\}$. Then $\hat{L}$, $\hat{\boldsymbol{h}}_{\mathbb{Z}}$, h, z^h ($h \in \boldsymbol{h}$) act naturally on V_L by acting on either $M(1)$ or $\mathbb{C}\{L\}$ as indicated above. In particular, c acts as 1 and $\boldsymbol{h}_*$ acts trivially. The automorphism $\hat{\nu}$ acts in a natural grading-preserving way on V_L, via $\nu\otimes\hat{\nu}$, and this action is compatible with the other actions:

$$\hat{\nu}(a\cdot v) = \hat{\nu}(a)\cdot\hat{\nu}(v), \tag{3.23}$$

$$\hat{\nu}(u\cdot v) = \nu(u)\cdot\hat{\nu}(v), \tag{3.24}$$

$$\hat{\nu}(z^h\cdot v) = z^{\nu(h)}\cdot\hat{\nu}(v) \tag{3.25}$$

for $a \in \hat{L}, u \in \hat{\boldsymbol{h}}, h \in \boldsymbol{h}, v \in V_L$.

It will be convenient later to have the notation

$$c(\alpha, \beta) = \omega_q^{c_0(\alpha,\beta)} \in \mathbb{C}^\times \tag{3.26}$$

for all $\alpha, \beta \in L$. Then

$$aba^{-1}b^{-1} = c(\bar{a}, \bar{b}) \tag{3.27}$$

for $a, b \in \hat{L}$, as operators on $\mathbb{C}\{L\}$ and on V_L, and

$$c(\nu\alpha, \nu\beta) = c(\alpha, \beta). \tag{3.28}$$

For $\alpha \in \boldsymbol{h}$, $n \in \mathbb{Z}$, we write $\alpha(n)$ for the operator on V_L determined by $\alpha \otimes t^n$. Set

$$\Omega_* = \{v \in V_L \mid h(n)v = 0 \text{ for } h \in \boldsymbol{h}_*, n > 0\}, \tag{3.29}$$

$$V_* = \operatorname{span}\{h(n)V_L \mid h \in \boldsymbol{h}_*, n < 0\}. \tag{3.30}$$

Then Ω_* is the vacuum space for the Heisenberg algebra $(\hat{\boldsymbol{h}}_*)_{\mathbb{Z}}$ (defined as in (3.7)) and we have

$$V_L = \Omega_* \oplus V_*. \tag{3.31}$$

In fact, we see from (3.21) that V_L has the decomposition

$$V_L = S(\hat{\boldsymbol{h}}_*^-) \otimes S((\hat{\boldsymbol{h}}_*^\perp)^-) \otimes \mathbb{C}\{L\} \tag{3.32}$$

and also

$$\Omega_* = S((\hat{\boldsymbol{h}}_*^\perp)^-) \otimes \mathbb{C}\{L\}, \tag{3.33}$$

$$V_* = \hat{\boldsymbol{h}}_*^- S(\hat{\boldsymbol{h}}_*^-) \otimes \Omega_*. \tag{3.34}$$

Here $\hat{\boldsymbol{h}}_*^-$ and $(\hat{\boldsymbol{h}}_*^\perp)^-$ are defined as in (3.6). In terms of the general structure of modules for Heisenberg algebras, we know from [14, Section 1.7], that the (well-defined) canonical linear map

$$\begin{aligned} U((\hat{\boldsymbol{h}}_*)_{\mathbb{Z}}) \otimes_{(\hat{\boldsymbol{h}}_*^+ \oplus \mathbb{C}c)} \Omega_* &\to V_L \\ u \otimes v &\mapsto u \cdot v \end{aligned} \tag{3.35}$$

($u \in U((\hat{\boldsymbol{h}}_*)_{\mathbb{Z}}), v \in \Omega_*$) is an $(\hat{\boldsymbol{h}}_*)_{\mathbb{Z}}$-module isomorphism, and in particular, that the linear map

$$\begin{aligned} M_*(1) \otimes_{\mathbb{C}} \Omega_* = U(\hat{\boldsymbol{h}}_*^-) \otimes_{\mathbb{C}} \Omega_* &\to V_L \\ u \otimes v &\mapsto u \cdot v \end{aligned} \tag{3.36}$$

($u \in U(\hat{\boldsymbol{h}}_*^-), v \in \Omega_*$) is an $(\hat{\boldsymbol{h}}_*)_{\mathbb{Z}}$-module isomorphism, Ω_* now being regarded as a trivial $(\hat{\boldsymbol{h}}_*)_{\mathbb{Z}}$-module, where $M_*(1)$ is the $(\hat{\boldsymbol{h}}_*)_{\mathbb{Z}}$-module defined by analogy with (3.8). The spaces Ω_* and V_* are $\mathbb{C}$-graded and are stable under the actions of $\hat{\boldsymbol{h}}_*^\perp$ (defined as in (3.1)), of $\hat{L}$ and of $\hat{\nu}$.

For $\alpha \in \boldsymbol{h}$, set

$$\alpha(z) = \sum_{n \in \mathbb{Z}} \alpha(n) z^{-n-1}. \tag{3.37}$$

We use a normal ordering procedure, indicated by open colons, which signify that the enclosed expression is to be reordered if necessary so that all the operators $\alpha(n)$ ($\alpha \in \boldsymbol{h}$, $n < 0$), $a \in \hat{L}$ are to be placed to the left of all the operators $\alpha(n)$, z^α($\alpha \in \boldsymbol{h}, n \geq 0$) before the expression is evaluated. For $a \in \hat{L}$, set

$$Y_*(a, z) = {}^\circ_\circ e^{\int (\bar{a}'(z) - \bar{a}'(0) z^{-1})} a z^{\bar{a}'} {}^\circ_\circ, \tag{3.38}$$

using an obvious formal integration notation. (Note that the symbol $z^{\bar{a}'}$ could be replaced by $z^{\bar{a}}$ in this formula, in view of (3.19)). Let

$$a \in \hat{L}, \quad \alpha, \ldots, \alpha_k \in \boldsymbol{h}, \quad n_1, \ldots, n_k \in \mathbb{Z} \ (n_i > 0)$$

and set

$$\begin{aligned} v &= \alpha_1(-n_1) \cdots \alpha_k(-n_k) \otimes \iota(a) \\ &= \alpha_1(-n_1) \cdots \alpha_k(-n_k) \cdot \iota(a) \in V_L. \end{aligned} \tag{3.39}$$

We define

$$Y_*(v,z) = {}_\circ^\circ \left(\frac{1}{(n_1-1)!} \left(\frac{\mathrm{d}}{\mathrm{d}z} \right)^{n_1-1} \alpha_1'(z) \right) \cdots \left(\frac{1}{(n_k-1)!} \left(\frac{\mathrm{d}}{\mathrm{d}z} \right)^{n_k-1} \alpha_k'(z) \right) Y_*(a,z) {}_\circ^\circ. \tag{3.40}$$

This gives us a well-defined linear map

$$\begin{aligned} &V_L \to (\mathrm{End}\, V_L)\{z\} \\ &v \mapsto Y_*(v,z) = \sum_{n \in \mathbb{C}} v_n z^{-n-1}, \quad v_n \in \mathrm{End}\, V_L, \end{aligned} \tag{3.41}$$

where for any vector space W, we define $W\{z\}$ to be the vector space of W-valued formal series in z, with arbitrary complex powers of z allowed:

$$W\{z\} = \left\{ \sum_{n \in \mathbb{C}} w_n z^n \mid w_n \in W \right\}. \tag{3.42}$$

We call $Y_*(v,z)$ the *untwisted vertex operator associated with* v*, defined relative to* $\boldsymbol{h}_*$. We shall be especially interested in these operators for $v \in \Omega_*$. Note that the component operators v_n of $Y_*(v,z)$ are defined in (3.41).

Remark 3.2. The case $\boldsymbol{h}_* = 0$ recovers the case of ordinary untwisted vertex operators as defined in [2, 14]. In this case, $\Omega_* = V_L$, $V_* = 0$, and Y_* is the operator Y, in the notation of [14].

It is easy to check from the definition that the relative untwisted vertex operators $Y_*(v,z)$ for $v \in V_L$ have the following properties:

$$Y_*(1,z) = 1, \tag{3.43}$$

$$Y_*(v,z) = 0 \quad \text{if } v \in V_*, \tag{3.44}$$

$$Y_*(v,z)\iota(1) \in V_L[[z]] \text{ and } \lim_{z \to 0} Y_*(v,z)\iota(1) = v \qquad \text{if } v \in \Omega_*, \tag{3.45}$$

$$[(\hat{\boldsymbol{h}}_*)_{\mathbb{Z}}, Y_*(v,z)] = 0, \tag{3.46}$$

$$Y_*(a \cdot v, z) = a Y_*(v,z) \tag{3.47}$$

for $a \in \hat{L}$ such that $\bar{a} \in L \cap \boldsymbol{h}_*$. (For a vector space W, $W[[z]]$ signifies the space of formal power series in z with coefficients in W.) Moreover, using (3.23)–(3.25) we see that

$$\hat{\nu} Y_*(v,z) \hat{\nu}^{-1} = Y_*(\hat{\nu} v, z). \tag{3.48}$$

By (3.46), the component operators v_n of $Y_*(v,z)$ preserve Ω_* and V_*, so that we get a well-defined linear map

$$\begin{aligned} &V_L \to (\operatorname{End} \Omega_*)\{z\} \\ &v \mapsto Y_*(v,z). \end{aligned} \tag{3.49}$$

We continue to denote the restriction of this map to Ω_* by Y_*:

$$\begin{aligned} &\Omega_* \to (\operatorname{End} \Omega_*)\{z\} \\ &v \mapsto Y_*(v,z). \end{aligned} \tag{3.50}$$

A basic property of the operators (3.38) and ultimately, of the operators (3.40), is: For $a, b \in \hat{L}$,

$$Y_*(a,z_1) Y_*(b,z_2) = {}^{\circ}_{\circ} Y_*(a,z_1) Y_*(b,z_2) {}^{\circ}_{\circ} (z_1 - z_2)^{\langle \bar{a}', \bar{b}' \rangle}, \tag{3.51}$$

where the binomial expression is to be expanded in nonnegative integral powers of the second variable, z_2.

Remark 3.3. In our frequent use of formal series, it will typically be understood that a binomial expression such as that in (3.51) is to be expanded as in (3.51) – as a formal power series in the second variable.

4. Relative twisted vertex operators

In this section, we systematically construct a setting which generalizes that of Section 3 to incorporate the lattice-isometry ν. Many definitions and notations in Section 3 have natural generalizations in this setting. We shall define a twisted space V_L^T and present a notion of relative twisted vertex operator $Y_*^{\nu}(v,z)$ $(v \in V_L)$ acting on V_L^T. We shall point out that the twisted vertex operators introduced in [19, 13] are special cases of our setting.

For $n \in \mathbb{Z}$, set

$$\boldsymbol{h}_{(n)} = \{\alpha \in \boldsymbol{h} \mid \nu\alpha = \omega_p^n \alpha\} \subset \boldsymbol{h}, \tag{4.1}$$

where $\omega_p = \omega_q^{q/p}$, which is primitive pth root of unity in $\mathbb{C}$ (recall (3.11)). Then

$$\boldsymbol{h} = \coprod_{n \in \mathbb{Z}/p\mathbb{Z}} \boldsymbol{h}_{(n)} \tag{4.2}$$

(we identify $\boldsymbol{h}_{(n \bmod p)}$ with $\boldsymbol{h}_{(n)}$ for $n \in \mathbb{Z}$). For $\alpha \in \boldsymbol{h}$ write $\alpha_{(n)}$ for the component of α in $\boldsymbol{h}_{(n)}$. We define the ν-twisted affine Lie algebra $\hat{\boldsymbol{h}}[\nu]$ associated with the abelian Lie algebra $\boldsymbol{h}$ to be

$$\hat{\boldsymbol{h}}[\nu] = \coprod_{n \in (1/p)\mathbb{Z}} \boldsymbol{h}_{(pn)} \otimes t^n \oplus \mathbb{C}c \tag{4.3}$$

with

$$[x \otimes t^m, y \otimes t^n] = \langle x, y \rangle m \delta_{m+n,0} c \quad \text{for } x \in \boldsymbol{h}_{(pm)},\ y \in \boldsymbol{h}_{(pn)},\ m, n \in \frac{1}{p}\mathbb{Z}, \tag{4.4}$$

$$[c, \hat{\boldsymbol{h}}[\nu]] = 0. \tag{4.5}$$

As in Section 3, this notation and related notation below may be applied to any finite-dimensional abelian Lie algebra with a nonsingular symmetric bilinear form and with a finite-order isometry. Note that for the identity automorphism $\nu = 1$, with p chosen to be 1, the twisted algebra $\hat{\boldsymbol{h}}[\nu]$ reduces to the untwisted algebra $\hat{\boldsymbol{h}}$ of (3.1)–(3.3). Define the *weight gradation* on $\hat{\boldsymbol{h}}[\nu]$ associated with $\boldsymbol{h}_*$ by

$$\mathrm{wt}(x \otimes t^m) = 0, \qquad \mathrm{wt}(y \otimes t^n) = -n, \qquad \mathrm{wt}\, c = 0 \tag{4.6}$$

for $m, n \in (1/p)\mathbb{Z}$, $x \in (\boldsymbol{h}_*)_{(pm)}$, $y \in (\boldsymbol{h}_*^{\perp})_{(pn)}$ (cf. (3.5)). Set

$$\hat{\boldsymbol{h}}[\nu]^+ = \coprod_{n > 0} \boldsymbol{h}_{(pn)} \otimes t^n, \qquad \hat{\boldsymbol{h}}[\nu]^- = \coprod_{n < 0} \boldsymbol{h}_{(pn)} \otimes t^n. \tag{4.7}$$

Now the subalgebra

$$\hat{\boldsymbol{h}}[\nu]_{(1/p)\mathbb{Z}} = \hat{\boldsymbol{h}}[\nu]^+ \oplus \hat{\boldsymbol{h}}[\nu]^- \oplus \mathbb{C}c \tag{4.8}$$

of $\hat{\boldsymbol{h}}[\nu]$ is a Heisenberg algebra (cf. (3.7)). Form the induced $\hat{\boldsymbol{h}}[\nu]$-module

$$S[\nu] = U(\hat{\boldsymbol{h}}[\nu]) \otimes_{U(\coprod_{n \geq 0} \boldsymbol{h}_{(pn)} \otimes t^n \oplus \mathbb{C}c)} \mathbb{C} \simeq S(\hat{\boldsymbol{h}}[\nu]^-) \quad \text{(linearly)}, \tag{4.9}$$

which is irreducible under $\hat{\boldsymbol{h}}[\nu]_{(1/p)\mathbb{Z}}$, where $\coprod_{n \geq 0} \boldsymbol{h}_{(pn)} \otimes t^n$ acts trivially on $\mathbb{C}$ and c acts as 1. We give the module $S[\nu]$ a $\mathbb{Q}$-grading compatible with the action of $\hat{\boldsymbol{h}}[\nu]$, and such that

$$\mathrm{wt}\, 1 = \frac{1}{4p^2} \sum_{k=1}^{p-1} k(p-k) \dim(\boldsymbol{h}_*^{\perp})_{(k)} \tag{4.10}$$

(the reason for choosing this shifted gradation will be explained in Section 6). The lattice-isometry ν acts naturally on $\hat{\boldsymbol{h}}[\nu]$ (fixing c):

$$\nu(\alpha \otimes t^n) = \omega_p^n \alpha \otimes t^n$$

for $n \in (1/p)\mathbb{Z}$, $\alpha \in \boldsymbol{h}_{(pn)}$, and on $S[\nu]$ as an algebra isomorphism, preserving the gradation, and we have

$$\nu(u \cdot v) = v(u) \cdot \nu(u) \tag{4.11}$$

for $y \in \hat{\boldsymbol{h}}[\nu]$ and $v \in S[\nu]$. For $\alpha \in \boldsymbol{h}$, $n \in (1/p)\mathbb{Z}$, write $\alpha(n)$ for the operator on $S[\nu]$ determined by $\alpha_{(pn)} \otimes t^n$.

Let T be an $\hat{L}_\nu$-module with κ_q acting as multiplication by ω_q. We assume that $\boldsymbol{h}_{(0)}$ acts on T in such a way that

$$T = \coprod_{\alpha \in \boldsymbol{h}_{(0)} \cap \boldsymbol{h}_*^\perp} T_\alpha, \tag{4.12}$$

where

$$T_\alpha = \{t \in T \mid h \cdot t = \langle h, \alpha \rangle t \text{ for } h \in \boldsymbol{h}_{(0)}\}, \tag{4.13}$$

and such that the actions of $\hat{L}_\nu$ and $\boldsymbol{h}_{(0)}$ are compatible in the sense that

$$a \cdot T_\alpha \subset T_{\alpha + \bar{a}'_{(0)}} \tag{4.14}$$

for $a \in \hat{L}_\nu$ and $\alpha \in \boldsymbol{h}_{(0)} \cap \boldsymbol{h}_*^\perp$. We also assume that $\hat{\nu}$ acts on T as a linear automorphism such that

$$\hat{\nu} T_\alpha \subset T_{\nu\alpha}. \tag{4.15}$$

Then as operators on T,

$$ha = a(\langle h', \bar{a} \rangle + h), \tag{4.16}$$

$$\hat{\nu} h \hat{\nu}^{-1} = h \tag{4.17}$$

for $h \in \boldsymbol{h}_{(0)}$ and $a \in \hat{L}_\nu$. Define a $\mathbb{C}$-gradation on T by

$$\operatorname{wt} t = \frac{1}{2} \langle \alpha', \alpha' \rangle \quad \text{for } t \in T_{\alpha'} \ (\alpha \in \boldsymbol{h}_{(0)}). \tag{4.18}$$

Then $\hat{\nu}$ preserves this gradation of T by (4.15). We define an End T-valued formal Laurent series z^h for $h \in \boldsymbol{h}_{(0)}$ as follows:

$$z^h \cdot t = z^{\langle h, \alpha \rangle} t \quad \text{for } t \in T_\alpha \ (\alpha \in \boldsymbol{h}_{(0)} \cap \boldsymbol{h}_*^\perp). \tag{4.19}$$

Then from (4.16),

$$z^h a = a z^{\langle h', \bar{a} \rangle + h} \quad \text{for } a \in \hat{L}_\nu \tag{4.20}$$

as operators on T.

Remark 4.1. If $\nu = 1$ and $c_0(\cdot,\cdot) = c_0^\nu(\cdot,\cdot)$, then $\hat{L}_\nu = \hat{L}$ and $\mathbb{C}\{L\}$ is a such module T.

Remark 4.2. In the setting of Remark 2.2, let L be an even lattice satisfying (2.8) with the alternating bilinear map c_0^ν given in (2.10). Then $K = \{a^{-1}\hat{\nu}(a) \mid a \in \hat{L}_\nu\}$ is a central subgroup of $\hat{L}_\nu$ and $K \cap \langle \kappa_q \rangle = 1$. In [19], a certain class of $\hat{L}_\nu$-modules on which κ_q acts by ω_q and K acts according to the character $\chi(a^{-1}\hat{\nu}a) = \omega_p^{-\langle \sum \nu^r \bar{a}, \bar{a} \rangle / 2}$ is classified and constructed explicitly (see Propositions 6.1 and 6.2 of [19]). These modules have the properties described above with $\boldsymbol{h}_*^\perp = \boldsymbol{h}$. In particular, any such module T (which is denoted by U_T in [19]) has the following decomposition:

$$T = \sum_{a \in L} T_{\bar{a}_{(0)}}. \tag{4.21}$$

For $\alpha \in \boldsymbol{h}_{(0)}$, if $\langle \alpha, \bar{a}_{(0)} \rangle \in \mathbb{Z}$ for all $a \in L$, define the operator ω_q^α on T by

$$\omega_q^\alpha \cdot t = \omega_q^{\langle \alpha, \bar{a}_{(0)} \rangle} t \tag{4.22}$$

for $t \in T_{\bar{a}_{(0)}}$. Then as operators on T,

$$\hat{v} a = a \omega_p^{-p\bar{a}_{(0)} - p\langle \bar{a}_{(0)}, \bar{a}_{(0)} \rangle/2}, \tag{4.23}$$

where $a \in \hat{L}_\nu$ and the operator on the right-hand side is well defined because of the assumption (2.8). It follows that

$$\hat{v}^p = 1 \tag{4.24}$$

(see [19]), a nontrivial fact that is not an automatic consequence of the assumption $\nu^p = 1$.

Set

$$V_L^T = S[\nu] \otimes T \tag{4.25}$$

which is naturally graded, using the gradations of $S[\nu]$ and T. Again $\hat{L}_\nu$, $\hat{\boldsymbol{h}}[\nu]_{(1/p)\mathbb{Z}}$, $\boldsymbol{h}_{(0)}$, $z^h (h \in \boldsymbol{h}_{(0)})$ act on V_L^T by acting on either $S[\nu]$ on T. Then $\hat{\nu}$ extends to a linear automorphism of V_L^T so that $\hat{\nu}(u \otimes t) = \nu(u) \otimes \hat{\nu}(t)$ for $u \in S[\nu]$ and $t \in T$. As in Section 3, we write

$$c_\nu(\alpha, \beta) = \omega_q^{c_0^\nu(\alpha, \beta)} \tag{4.26}$$

for $\alpha, \beta \in L$. Then for $a, b \in \hat{L}_\nu$

$$aba^{-1}b^{-1} = c_\nu(\bar{a}, \bar{b}) \tag{4.27}$$

as operators on T and on V_L^T. We define two subspaces of V_L^T:

$$\Omega_*^\nu = \left\{ v \in V_L^T \mid h(n)v = 0, \text{ for } h \in \boldsymbol{h}_*, \ n \in \frac{1}{p}\mathbb{Z}, \ n > 0 \right\}, \tag{4.28}$$

$$V_*^\nu = \left\{ h(n) V_L^T \mid h \in \boldsymbol{h}_*, \ n \in \frac{1}{p}\mathbb{Z}, \ n < 0 \right\} \tag{4.29}$$

(cf. (3.29) and (3.30)). Then Ω_*^ν is the vacuum space for the Heisenberg algebra $\boldsymbol{h}_*[\nu]_{(1/p)\mathbb{Z}}$ and

$$V_L^T = V_*^\nu \oplus \Omega_*^\nu. \tag{4.30}$$

As in (3.33) and (3.34) we have

$$\Omega_*^\nu = S((\hat{\boldsymbol{h}}_*^\perp[\nu])^-) \otimes T, \tag{4.31}$$

$$V_*^\nu = \hat{\boldsymbol{h}}_*[\nu]^- S(\hat{\boldsymbol{h}}_*[\nu]^-) \otimes \Omega_*^\nu. \tag{4.32}$$

The linear map $\hat{\nu}$ preserves both Ω_*^ν and V_*.

For a nonzero complex number a we shall fix the branch of $a^\tau = e^{\tau(\log|a| + i\arg a)}$ (τ is a complex variable) with $-\pi < \arg a \le \pi$. Then $a^{\tau+\zeta} = a^\tau a^\zeta$ for $\tau, \zeta \in \mathbb{C}$. The following lemma will be used in proving the Jacobi identity in the next section:

Lemma 4.3. *For* $\tau \in \mathbb{C}$,

$$\prod_{r=1}^{p-1} (1 - \omega_p^r)^\tau = p^\tau. \tag{4.33}$$

Proof. If $\omega_p^r \neq -1$, $\arg(1 - \omega_p^r) = -\arg(1 - \omega_p^{-r})$. Then

$$(1 - \omega_p^r)^\tau (1 - \omega_p^{-r})^\tau = ((1 - \omega_p^r)(1 - \omega_p^{-r}))^\tau$$

and the lemma follows from the fact that $\prod_{r=1}^{p-1}(1 - \omega_p^r) = p$. □

For $\alpha \in L$, we define

$$\sigma(\alpha) = \begin{cases} \prod_{0<r<p/2} (1 - \omega_p^{-r})^{\langle \nu^r\alpha', \alpha'\rangle} 2^{\langle \nu^{p/2}\alpha', \alpha'\rangle} & \text{if } p \in 2\mathbb{Z} \\ \prod_{0<r<p/2} (1 - \omega_p^{-r})^{\langle \nu^r\alpha', \alpha'\rangle} & \text{if } p \in 2\mathbb{Z}+1. \end{cases} \tag{4.34}$$

Then $\sigma(\nu\alpha) = \sigma(\alpha)$. We also define

$$\varepsilon_1(\alpha, \beta) = \prod_{0<r<p} (1 - \omega_p^{-r})^{\langle \nu^r\alpha, \beta\rangle}, \tag{4.35}$$

$$\varepsilon_2(\alpha, \beta) = \omega_q^{\varepsilon_0(\alpha, \beta)} \tag{4.36}$$

for $\alpha, \beta \in L$.

Remark 4.4. In the situation of Remark 2.2, the functions $\sigma(\alpha), \varepsilon_1(\alpha, \beta)$ and $\varepsilon_2(\alpha, \beta)$ were introduced in [19] with $\boldsymbol{h}_*^\perp = \boldsymbol{h}$ to define general twisted vertex operators. One can check that in this case

$$\varepsilon_1(\alpha, \beta) = \varepsilon_2(\alpha, \beta)\sigma(\alpha + \beta)/\sigma(\alpha)\sigma(\beta) \tag{4.37}$$

for $\alpha, \beta \in L$.

For $\alpha \in \boldsymbol{h}$, set

$$\alpha(z) = \sum_{n \in (1/p)\mathbb{Z}} \alpha(n) z^{-n-1}. \tag{4.38}$$

We now define the *relative ν-twisted vertex operator* $Y_*^\nu(a, z)$ for $a \in \hat{L}$ acting on V_L^T as follows:

$$Y_*^\nu(a, z) = p^{-\langle \bar{a}', \bar{a}'\rangle/2} \sigma(\bar{a}) {}^\circ_\circ e^{\int(\bar{a}'(z) - \bar{a}'(0)z^{-1})} a z^{\bar{a}'_{(0)} + \langle \bar{a}'_{(0)}, \bar{a}'_{(0)}\rangle/2 - \langle \bar{a}', \bar{a}'\rangle/2} {}^\circ_\circ \tag{4.39}$$

generalizing the relative untwisted vertex operator $Y_*(a, z)$, the case $\nu = 1$. Note that on the right-hand side, we view a as an element of $\hat{L}_\nu$, according to our set-theoretic identification between $\hat{L}$ and $\hat{L}_\nu$ (see Remark 2.1). The numerical factor at the front of (4.39) leads to just the right form for the general result below.

For $\alpha_1, \dots, \alpha_k \in \boldsymbol{h}$, $n_1, \dots, n_k \in \mathbb{Z}$ $(n_i > 0)$ and $v = \alpha_1(-n_1) \cdots \alpha_k(-n_k) \cdot \iota(a) \in V_L$, set

$$W_*(v,z) = {}^{\circ}_{\circ}\left(\frac{1}{(n_1-1)!}\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{n_1-1}\alpha_1'(z)\right)\cdots\left(\frac{1}{(n_k-1)!}\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{n_k-1}\alpha_k'(z)\right)Y_*^\nu(a,z){}^{\circ}_{\circ}, \tag{4.40}$$

where the right side is an operator on V_L^T. Extend to all $v \in V_L$ by linearity.

Let $\{\beta_1, \dots, \beta_d\}$ be an orthonormal basis of $\boldsymbol{h}_*^\perp$, and define constants $c_{mni} \in \mathbb{C}$ for $m, n \geq 0$ and $i = 0, \dots, p-1$ by the formulas

$$\sum_{m,n\geq 0} c_{mn0}x^m y^n = -\frac{1}{2}\sum_{r=1}^{p-1}\log\left(\frac{(1+x)^{1/p} - \omega_p^{-r}(1+y)^{1/p}}{1-\omega_p^{-r}}\right),$$

$$\sum_{m,n\geq 0} c_{mni}x^m y^n = \frac{1}{2}\log\left(\frac{(1+x)^{1/p} - \omega_p^{-i}(1+y)^{1/p}}{1-\omega_p^{-i}}\right) \quad \text{for } i \neq 0. \tag{4.41}$$

Set

$$\Delta_z = \sum_{m,n\geq 0}\sum_{i=0}^{p-1}\sum_{j=1}^{d} c_{mni}(\nu^{-i}\beta_j)(m)\beta_j(n)z^{-m-n}. \tag{4.42}$$

Then e^{Δ_z} is well defined on V_L since $c_{00i} = 0$ for all i, and for $v \in V_L$, $\mathrm{e}^{\Delta_z}v \in V_L[z^{-1}]$. Note that Δ_z is independent of the choice of orthonormal basis. Then from (3.48),

$$\hat{\nu}\Delta_z = \Delta_z\hat{\nu} \tag{4.43}$$

and hence

$$\hat{\nu}\mathrm{e}^{\Delta_z} = \mathrm{e}^{\Delta_z}\hat{\nu} \tag{4.44}$$

on V_L. For $v \in V_L$, the *relative ν-twisted vertex operator* $Y_*^\nu(v,z)$ is defined by

$$Y_*^\nu(v,z) = W_*(\mathrm{e}^{\Delta_z}v, z). \tag{4.45}$$

Then this yields a well-defined linear map

$$\begin{aligned} &V_L \to (\mathrm{End}\, V_L^T)\{z\} \\ &v \mapsto Y_*^\nu(v,z) = \sum_{n\in\mathbb{C}} v_n^\nu z^{-n-1} \quad (v_n^\nu \in \mathrm{End}\, V_L^T). \end{aligned} \tag{4.46}$$

We now discuss the relation between ν and twisted vertex operators $Y_*^\nu(v,z)$ for $v \in V_L$. It is easy to see from the definitions (4.39) and (4.40) that

$$\nu\alpha(n)\nu^{-1} = (\nu\alpha)(n) \quad \text{for } \alpha \in \boldsymbol{h},\ n \in \frac{1}{p}\mathbb{Z}, \tag{4.47}$$

$$\nu Y_*^\nu(a,z)\nu^{-1} = Y_*^\nu(\hat{\nu}a, z) \quad \text{for } a \in \hat{L}, \tag{4.48}$$

$$\nu W_*(v,z)\nu^{-1} = W_*(\hat{\nu}v, z) \quad \text{for } v \in V_L \tag{4.49}$$

and it follows from (4.44) and (4.45) that

$$\nu Y_*^\nu(v,z)\nu^{-1} = Y_*^\nu(\hat{\nu}v,z) \quad \text{for } v \in V_L. \tag{4.50}$$

In particular, ν commutes with $Y_*^\nu(v,z)$ if v is $\hat{\nu}$-invariant. By the properties of $\hat{\nu}$ and its action on V_L^T, we also have

$$Y_*^\nu(\hat{\nu}^r v,z) = \lim_{z^{1/p}\to\omega_p^{-r}z^{1/p}} Y_*^\nu(v,z)\omega_p^{pr\bar{a}'_{(0)}+pr\langle\bar{a}'_{(0)},\bar{a}'_{(0)}\rangle/2-pr\langle\bar{a}',\bar{a}'\rangle/2}a^{-1}\hat{\nu}^r a \tag{4.51}$$

for $v = v'\otimes\iota(a)\in V_L$, where $v'\in S(\hat{\boldsymbol{h}}^-), a\in\hat{L}$ and the operator ω_p^α is defined as in (4.22). Note that ω_p^τ is well defined for any complex number τ.

Remark 4.5. In the context of Remark 2.2, let T be an $\hat{L}_\nu$-module as given in Remark 4.2. Then the operators $Y_*^\nu(v,z)$ are exactly the ν-twisted vertex operators $Y_\nu(v,z)$ introduced in [13]. In particular, $Y_\nu(u,z)\in(\operatorname{End}V_L^T)[[z^{1/p},z^{-1/p}]]$ for $u\in V_L$ and (4.51) reduces to

$$Y_\nu(\hat{\nu}^r v,z) = \lim_{z^{1/p}\to\omega_p^{-r}z^{1/p}} Y_\nu(v,z) \tag{4.52}$$

(see (4.23)).

We summarize the main elementary properties of relative twisted vertex operators in the following proposition.

Proposition 4.6. *Let $a,b\in\hat{L}, v\in V_L, \alpha\in\boldsymbol{h}_{(0)}, \psi\in\hat{L}$ such that $\bar{\psi}\in L\cap\boldsymbol{h}_*$. We have*

$$\overset{\circ}{\circ} Y_*^\nu(a,z_1)Y_*^\nu(b,z_2)\overset{\circ}{\circ} = c_\nu(a,b)\overset{\circ}{\circ} Y_*^\nu(b,z_2)Y_*^\nu(a,z_1)\overset{\circ}{\circ}, \tag{4.53}$$

$$Y_*^\nu(a,z_1)Y_*^\nu(b,z_2) = \overset{\circ}{\circ} Y_*^\nu(a,z_1)Y_*^\nu(b,z_2)\overset{\circ}{\circ}\prod_{i=0}^{p-1}(z_1^{1/p}-\omega_p^{-i}z_2^{1/p})^{\langle\nu^i\bar{a}',\bar{b}'\rangle}, \tag{4.54}$$

$$\begin{aligned}\frac{d}{dz}Y_*^\nu(a,z) &= \overset{\circ}{\circ}\left(\bar{a}'(z)+\frac{1}{2}\langle\bar{a}'_{(0)},\bar{a}'_{(0)}\rangle z^{-1}-\frac{1}{2}\langle\bar{a},\bar{a}\rangle z^{-1}\right)Y_*^\nu(a,z)\overset{\circ}{\circ}\\ &= Y_*^\nu(\bar{a}(-1)a,z),\end{aligned} \tag{4.55}$$

$$Y_*^\nu(v,z) = 0 \quad \text{if } v\in V_*, \tag{4.56}$$

$$[\hat{\boldsymbol{h}}_*[\nu]_{(1/p)\mathbb{Z}}, Y_*^\nu(v,z)] = 0 \tag{4.57}$$

and v_n^ν preserves both Ω_*^ν and V_*^ν for $n\in\mathbb{C}$,

$$Y_*^\nu(\psi v,z) = \omega_q^{\varepsilon_0(\bar{\psi},\bar{a})}\psi Y_*^\nu(v,z) \tag{4.58}$$

if $v = v^*\otimes\iota(a)$ for $v^*\in S(\hat{\boldsymbol{h}}^-)$ (see (2.4)), and in particular

$$Y_*^\nu(\iota(\psi),z) = \psi. \tag{4.59}$$

(The binomial expressions in (4.54) are to be expanded in nonnegative integral powers of the second variable $z_2^{1/p}$; recall Remark 3.3.)

Proof. First, (4.53) follows from the facts that

$$\,{}^{\circ}_{\circ}Y^{\nu}_{*}(a,z_1)\,Y^{\nu}_{*}(b,z_2){}^{\circ}_{\circ} = p^{-\langle \bar{a}',\bar{a}'\rangle/2-\langle \bar{b}',\bar{b}'\rangle/2}\sigma(\bar{a})\sigma(\bar{b}){}^{\circ}_{\circ}e^{\int(\bar{a}'(z_1)-\bar{a}'(0)z_1^{-1}+\bar{b}'(z_2)-\bar{b}'(0)z_2^{-1})}{}^{\circ}_{\circ}$$
$$\times\, ab z_1^{\bar{a}'_{(0)}+\langle \bar{a}'_{(0)},\bar{a}'_{(0)}\rangle/2-\langle \bar{a}',\bar{a}'\rangle/2} z_2^{\bar{b}'_{(0)}+\langle \bar{b}'_{(0)},\bar{b}'_{(0)}\rangle/2-\langle \bar{b}',\bar{b}'\rangle/2} \tag{4.60}$$

and that $ab = c_\nu(a,b)ba$. (4.54) follows from (4.20) and

$$e^{\int(\bar{a}'(z_1)-\bar{a}'(0)z_1^{-1}+\bar{b}'(z_2)-\bar{b}'(0)z_2^{-1})} = {}^{\circ}_{\circ}e^{\int(\bar{a}'(z_1)-\bar{a}'(0)z_1^{-1}+\bar{b}'(z_2)-\bar{b}'(0)z_2^{-1})}{}^{\circ}_{\circ}$$
$$\times \prod_{i=0}^{p-1}\left(1-\omega_p^{-i}\left(\frac{z_2}{z_1}\right)^{1/p}\right)^{\langle \nu^i\bar{a}',\bar{b}'\rangle},$$

and (4.55) follows from the computation

$$e^{\Delta_z}\bar{a}(-1)a = \left(\bar{a}'(-1)+\frac{1}{2}\langle \bar{a}'_{(0)},\bar{a}'_{(0)}\rangle z^{-1}-\frac{1}{2}\langle \bar{a}',\bar{a}'\rangle z^{-1}\right)a$$

and the definition of relative twisted vertex operators (4.45). The other relations are the direct consequences of the definitions (4.39), (4.40) and (4.45). □

Since $Y^{\nu}_{*}(u,z)=0$ for $u\in V_*$ we shall be especially interested in the operators $Y^{\nu}_{*}(v,z)$ for $v\in\Omega_*$. We also use the notation Y^{ν}_{*} for the restriction map

$$\begin{aligned}\Omega_* &\to (\mathrm{End}\,V_L^T)\{z\}\\ v &\mapsto Y^{\nu}_{*}(v,z)=\sum_{n\in\mathbb{C}} v^{\nu}_n z^{-n-1}.\end{aligned} \tag{4.61}$$

5. A Jacobi identity for the relative twisted vertex operators

We continue our discussion of relative vertex operators. Here we present the main theorem – a Jacobi identity for relative twisted vertex operators, whose proof closely follows the pattern of the proof of the corresponding result in Ch. 9 of [14]. This result generalizes a large number of known ones. The case $\nu = 1$ amounts to the theory of relative untwisted vertex operators of [6, 7] in which we clarified the essential equivalence between untwisted Z-algebras [22] and parafermion algebras [32]. If the lattice is even and $\boldsymbol{h}_* = 0$, we recover the twisted vertex operators and the associated results obtained in [19, 13]. In certain special cases we also recover the Z-algebra structure and relations found in [24–28, 21–23]. In the case that L is a direct sum of several copies of the root lattice of the Lie algebra $\mathrm{sl}(2,\mathbb{C})$ and $\nu = -1$, the Jacobi identity for certain relative twisted vertex operators was established in [16] to explain the essential equivalence between the "twisted Z-algebras" associated with the principal representations of $A_1^{(1)}$ [26, 27] and the "twisted parafermion algebras" of [33].

We shall use the basic generating function

$$\delta(z)=\sum_{n\in\mathbb{Z}} z^n, \tag{5.1}$$

formally the expansion of the δ-function at $z = 1$. The fundamental (and elementary) properties of the δ-function can be found in [14]. The following result concerning formal calculus and the "δ-function" will be used in the proof of the Jacobi identity:

Proposition 5.1. *Let V be a vector space, $a(m,n)$ $(m,n \in (1/p)\mathbb{Z})$ a family of operators on V. Let $s,t \in \mathbb{C}$ and let ζ be a pth root of unity. Consider the formal series*

$$A(z_1^{1/p}, z_2^{1/p}) = z_1^s z_2^t \left(\sum_{m,n \in (1/p)\mathbb{Z}} a(m,n) z_1^m z_2^n \right) \tag{5.2}$$

and assume that

$$\lim_{z_1^{1/p} \to z_2^{1/p}} A(z_1^{1/p}, z_2^{1/p}) \quad \textit{exists,} \tag{5.3}$$

that is, for every $k \in \mathbb{Q}$ and $v \in V$, $a(m, k-m)v = 0$ for all but a finite number of $m \in (1/p)\mathbb{Z}$. Then

$$\begin{aligned} &A(z_1^{1/p}, z_2^{1/p}) e^{z_0(\partial/\partial z_1)} \delta\left(\zeta \left(\frac{z_1}{z_2}\right)^{1/p}\right) \\ &= \zeta^{sp} A(\zeta^{-1}(z_2 - z_0)^{1/p}, z_2^{1/p}) e^{z_0(\partial/\partial z_1)} \left(\left(\frac{z_1}{z_2}\right)^s \delta\left(\zeta\left(\frac{z_1}{z_2}\right)^{1/p}\right)\right) \\ &= (z_2 - z_0)^s z_2^t \left(\sum_{m,n \in (1/p)\mathbb{Z}} a(m,n)(z_2 - z_0)^m z_2^n \right) e^{z_0(\partial/\partial z_1)} \left(\left(\frac{z_1}{z_2}\right)^s \delta\left(\zeta\left(\frac{z_1}{z_2}\right)^{1/p}\right)\right). \end{aligned} \tag{5.4}$$

Proof. We have

$$\begin{aligned} &A(z_1^{1/p}, z_2^{1/p}) e^{z_0(\partial/\partial z_1)} \delta\left(\zeta \left(\frac{z_1}{z_2}\right)^{1/p}\right) \\ &= \left(\frac{z_1}{z_2}\right)^s \left(\frac{z_1}{z_2}\right)^{-s} A(z_1^{1/p}, z_2^{1/p}) e^{z_0(\partial/\partial z_1)} \delta\left(\zeta \left(\frac{z_1}{z_2}\right)^{1/p}\right) \\ &= \left(\frac{z_1}{z_2}\right)^s e^{z_0(\partial/\partial z_1)} \left(\left(\frac{z_1 - z_0}{z_2}\right)^{-s} A((z_1 - z_0)^{1/p}, z_2^{1/p}) \delta\left(\zeta\left(\frac{z_1}{z_2}\right)^{1/p}\right)\right) \\ &= \left(\frac{z_1}{z_2}\right)^s e^{z_0(\partial/\partial z_1)} \left(\zeta^{ps} \left(\frac{z_2 - z_0}{z_2}\right)^{-s} A(\zeta^{-1}(z_2 - z_0)^{1/p}, z_2^{1/p}) \delta\left(\zeta\left(\frac{z_1}{z_2}\right)^{1/p}\right)\right) \\ &= \zeta^{ps} A(\zeta^{-1}(z_2 - z_0)^{1/p}, z_2^{1/p}) \left(\frac{z_1}{z_2}\right)^s e^{z_0(\partial/\partial z_1)} \left(\left(1 - \frac{z_0}{z_2}\right)^{-s} \delta\left(\zeta\left(\frac{z_1}{z_2}\right)^{1/p}\right)\right) \\ &= \zeta^{ps} A(\zeta^{-1}(z_2 - z_0)^{1/p}, z_2^{1/p}) \left(\frac{z_1}{z_2}\right)^s e^{z_0(\partial/\partial z_1)} \left(\left(1 - \frac{z_0}{z_1}\right)^{-s} \delta\left(\zeta\left(\frac{z_1}{z_2}\right)^{1/p}\right)\right) \end{aligned}$$

$$= \zeta^{ps} A(\zeta^{-1}(z_2 - z_0)^{1/p}, z_2^{1/p}) \left(\frac{z_1}{z_2}\right)^s e^{z_0(\partial/\partial z_1)} \left(\left(\frac{z_1}{z_1 - z_0}\right)^s \delta\left(\zeta\left(\frac{z_1}{z_2}\right)^{1/p}\right)\right)$$

$$= \zeta^{ps} A(\zeta^{-1}(z_2 - z_0)^{1/p}, z_2^{1/p}) \left(\frac{z_1}{z_2}\right)^s \left(\frac{z_1 + z_0}{z_1}\right)^s e^{z_0(\partial/\partial z_1)} \delta\left(\zeta\left(\frac{z_1}{z_2}\right)^{1/p}\right)$$

$$= \zeta^{ps} A(\zeta^{-1}(z_2 - z_0)^{1/p}, z_2^{1/p}) e^{z_0(\partial/\partial z_1)} \left(\left(\frac{z_1}{z_2}\right)^s \delta\left(\zeta\left(\frac{z_1}{z_2}\right)^{1/p}\right)\right),$$

as desired. □

For notational convenience we write

$$F_{(\alpha,\beta)}(z_1, z_2) = \prod_{r=0}^{p-1} (z_1^{1/p} - \omega_p^{-r} z_2^{1/p})^{-\langle \nu^r \alpha', \beta' \rangle}, \tag{5.5}$$

$$G_{(\alpha,\beta)}(z_1, z_2) = \prod_{r=0}^{p-1} (z_1^{1/p} - \omega_p^{-r} z_2^{1/p})^{\langle \alpha', \beta' \rangle} F_{(\alpha,\beta)}(z_1, z_2) \tag{5.6}$$

for $\alpha, \beta \in L$, where as usual all binomial expressions are to be expanded in nonnegative integral powers of the second variable. We shall often use expressions like $F_{\alpha,\beta}(z_2 + z_0, z_2)$ and $G_{\alpha,\beta}(z_2 + z_0, z_2)$ below. It is understood that the expression $(z_2 + z_0)^{1/p} - \omega_p^r z_2^{1/p}$ is to be expanded in nonnegative integral powers of z_0, and so we have expansions of the form:

$$((z_2 + z_0)^{1/p} - \omega_p^r z_2^{1/p})^c = \begin{cases} p^{-c}(z_0/z_2)^c z_2^{c/p} \left(1 + \sum_{n \ge 1} a_n (z_0/z_2)^n\right) & \text{if } r = 0, \\ z_2^{c/p}(1 - \omega_p^r)^c \left(1 + \sum_{n \ge 1} b_n (z_0/z_2)^n\right) & \text{if } r > 0. \end{cases}$$

We also write

$$\tau(\alpha, \beta) = \frac{\sigma(\alpha)\sigma(\beta)\varepsilon_1(\alpha, \beta)}{\sigma(\alpha + \beta)\varepsilon_2(\alpha, \beta)} \in \mathbb{C} \tag{5.7}$$

for $\alpha, \beta \in L$. Note that in the context of Remark 2.2, $\alpha' = \alpha, \beta' = \beta$ and $\tau(\alpha, \beta) = 1$, by (4.37). The following general residue notation will be useful:

$$\mathrm{Res}_z\left(\sum v_n z^n\right) = v_{-1}. \tag{5.8}$$

Theorem 5.2. *Let $a, b \in \hat{L}$, $u^*, v^* \in M(1)$ and set*

$$u = u^* \otimes \iota(a) \in V_L, \qquad v = v^* \otimes \iota(b) \in V_L. \tag{5.9}$$

Then we have

$$F_{(\bar{a},\bar{b})}(z_1,z_2)(z_1-z_2)^n Y_*^\nu(u,z_1)Y_*^\nu(v,z_2)-c_\nu(\bar{a},\bar{b})$$
$$\times F_{(\bar{b},\bar{a})}(z_2,z_1)(-z_2+z_1)^n Y_*^\nu(v,z_2)Y_*^\nu(u,z_1)$$
$$=\mathrm{Res}_{z_0}\frac{1}{p}z_2^{-1}z_0^n\sum_{r=0}^{p-1}(G_{(\nu^r\bar{a},\bar{b})}(z_2+z_0,z_2)z_0^{-\langle \nu^r\bar{a}',\bar{b}'\rangle}Y_*^\nu(Y_*(\hat{\nu}^r u,z_0)v,z_2)\tau(\nu^r\bar{a},\bar{b})$$
$$\times c_\nu(\bar{a}-\nu^r\bar{a},\bar{b})(\hat{\nu}^r a^{-1})ae^{-z_0(\partial/\partial z_1)}\delta(\omega_p^r z_1^{1/p}/z_2^{1/p})(z_1/z_2)^{\bar{a}'_{(0)}+\langle \bar{a}'_{(0)},\bar{a}'_{(0)}\rangle/2-\langle \bar{a}',\bar{a}'\rangle/2}) \tag{5.10}$$

for $n\in\mathbb{Z}$, or equivalently,

$$z_0^{-1}F_{(\bar{a},\bar{b})}(z_1,z_2)\delta\left(\frac{z_1-z_2}{z_0}\right)Y_*^\nu(u,z_1)Y_*^\nu(v,z_2)$$
$$-c_\nu(\bar{a},\bar{b})z_0^{-1}F_{(\bar{b},\bar{a})}(z_2,z_1)\delta\left(\frac{z_2-z_1}{-z_0}\right)Y_*^\nu(v,z_2)Y_*^\nu(u,z_1)$$
$$=\frac{1}{p}z_2^{-1}\sum_{r=0}^{p-1}\left(G_{(\nu^r\bar{a},\bar{b})}(z_2+z_0,z_2)z^{-\langle \nu^r\bar{a}',\bar{b}'\rangle}Y_*^\nu(Y_*(\hat{\nu}^r u,z_0)v,z_2)\right.$$
$$\times\delta\left(\omega_p^r\left(\frac{z_1-z_0}{z_2}\right)^{1/p}\right)\tau(\hat{\nu}^r\bar{a},\bar{b})c_\nu(\bar{a}-\nu^r\bar{a},\bar{b})(\hat{\nu}^r a^{-1})a$$
$$\left.\times\left(\frac{z_1-z_0}{z_2}\right)^{\bar{a}'_{(0)}+\langle \bar{a}'_{(0)},\bar{a}'_{(0)}\rangle/2-\langle \bar{a}',\bar{a}'\rangle/2}\right), \tag{5.11}$$

where all binomial expressions are to be expanded in nonnegative integral powers of the second variables.

Proof. Let $k,l\geq 1$ and let $a_1,\dots,a_k,b_1,\dots,b_l\in\hat{L}$ subject to the conditions $a=a_1\cdots a_k$ and $b=b_1\cdots b_l$. Then the coefficients in the formal power series A and B,

$$A=\exp\left(\sum_{i=1}^k\sum_{n>0}\frac{\bar{a}'_i(-n)}{n}w_i^n\right)\iota(a)\in V_L[[w_1,\dots,w_k]],$$
$$B=\exp\left(\sum_{j=1}^l\sum_{n>0}\frac{\bar{b}'_j(-n)}{n}x_j^n\right)\iota(b)\in V_L[[x_1,\dots,x_l]], \tag{5.12}$$

span $S((\boldsymbol{h}_*^\perp)^-)\otimes\iota(a)$ and $S((\boldsymbol{h}_*^\perp)^-)\otimes\iota(b)$, respectively, and so it suffices to prove the theorem with u and v replaced by A and B, respectively.

Let $\alpha_1,\dots,\alpha_s\in L$ and $\alpha=\alpha_1+\cdots+\alpha_s$. For brevity, set

$$f_\alpha(z,w_1,\dots,w_s)=\prod_{1\leq i<j\leq s}\prod_{0\leq r<p}((z+w_i)^{1/p}-\omega_p^{-r}(z+w_j)^{1/p})^{\langle(\nu^r-1)\alpha'_i,\alpha'_j\rangle}, \tag{5.13}$$

$$f_{(\bar{a},\bar{b})}(z_1,w_1,\dots,w_k;z_2,x_1,\dots,x_l)$$
$$=\prod_{1\leq i\leq k,1\leq j\leq l}\prod_{0\leq r<p}((z+w_i)^{1/p}-\omega_p^{-r}(z_2+x_j)^{1/p})^{\langle \nu^r\bar{a}'_i,\bar{b}'_j\rangle}, \tag{5.14}$$

where all binomial expressions are to be understood as formal power series in the w's and x's, and the coefficient of each monomial in the w's in f_α is the product of $z^{\langle \bar{a}'_{(0)} - \bar{a}', \bar{b}\rangle}$ and a Laurent polynomial in z, and the coefficient of each monomial in the w's and x's in f_α is the product of $z_1^{\langle \bar{a}'_{(0)}, \bar{b}\rangle}$ with a Laurent polynomial in $z_1^{1/p}$ and $z_2^{1/p}$. We shall also use $f_{(\bar{b},\bar{a})}(z_2, x_1, \ldots, x_l; z_1, w_1, \ldots, w_k)$.

By (2.4), we see that as operators on V_L^T

$$a = \prod_{1 \le i < j \le k} \varepsilon_2(\bar{a}_i, \bar{a}_j) a_1 \cdots a_k. \tag{5.15}$$

Just as in the proof of Theorem 9.3.1 of [14], we have

$$\begin{aligned} W_*(A,z) &= {}^\circ_\circ \exp\left(\sum_{i=1}^{k} \sum_{n \ge 1} \frac{1}{n!} \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{n-1} \bar{a}'_i(z) w_i^n \right) Y_*^\nu(a,z) {}^\circ_\circ \\ &= {}^\circ_\circ Y_*^\nu(a_1, z + w_1) \cdots Y_*^\nu(a_k, z + w_k) {}^\circ_\circ \prod_{1 \le i < j \le k} p^{-\langle \bar{a}'_i, \bar{a}'_j\rangle} \varepsilon_2(\bar{a}_i, \bar{a}_j) \\ &\quad \times \frac{\sigma(\bar{a})}{\sigma(\bar{a}_1) \cdots \sigma(\bar{a}_k)} \prod_{1 \le i < j \le k} z^{\langle (\bar{a}'_i)_{(0)}, (\bar{a}'_j)_{(0)}\rangle - \langle \bar{a}'_i, \bar{a}'_j\rangle} \\ &\quad \times \prod_{1 \le i \le k} \left(1 + \frac{w_i}{z}\right)^{-\langle (\bar{a}'_i)_{(0)}, (\bar{a}'_i)_{(0)}\rangle/2 + \langle \bar{a}'_i, \bar{a}'_i\rangle/2} \end{aligned} \tag{5.16}$$

From (5.12),

$$A = {}^\circ_\circ Y_*(a_1, w_1) \cdots Y_*(a_k, w_k) {}^\circ_\circ \cdot \iota(1). \tag{5.17}$$

The action of e^{Δ_z} is given by

$$\begin{aligned} \mathrm{e}^{\Delta_z} A &= \prod_{i,j=1}^{k} \prod_{r=0}^{p-1} \exp\left(\sum_{s,t \ge 0, (s,t) \ne 0} c_{str} \langle \nu^r \bar{a}'_i, \bar{a}'_j\rangle \left(\frac{w_i}{z}\right)^s \left(\frac{w_j}{z}\right)^t \right) A \\ &= \prod_{i,j=1}^{k} \prod_{r=0}^{p-1} \exp\left(\sum_{s,t \ge 0, (s,t) \ne 0} c_{str} \left(\frac{w_i}{z}\right)^s \left(\frac{w_j}{z}\right)^t \right)^{\langle \nu^r \bar{a}'_i, \bar{a}'_j\rangle} A \\ &= \prod_{i,j=1}^{k} \prod_{r=1}^{p-1} \left(\frac{(1 - w_i/z)^{1/p} + \omega_p^{-r}(1 + w_j/z)^{1/p}}{1 - \omega_p^{-r}} \right)^{-\langle \bar{a}'_i, \bar{a}'_j\rangle/2 + \langle \nu^r \bar{a}'_i, \bar{a}'_j\rangle/2} A \\ &= \prod_{1 \le i < j \le k} \prod_{r=1}^{p-1} \left(\frac{(1 + w_i/z)^{1/p} - \omega_p^{-r}(1 + w_j/z)^{1/p}}{1 - \omega_p^{-r}} \right)^{-\langle \bar{a}'_i, \bar{a}'_j\rangle + \langle \nu^r \bar{a}'_i, \bar{a}'_j\rangle} \\ &\quad \times \prod_{1 \le i \le k} \left(1 + \frac{w_i}{z}\right)^{-\langle (\bar{a}'_i)_{(0)}, (\bar{a}'_i)_{(0)}\rangle/2 - \langle \bar{a}'_i, \bar{a}'_i\rangle/2} A, \end{aligned} \tag{5.18}$$

where these expansions are to be understood as the formal power series in the w's that they came from. From (5.16)–(5.18) we find that

$$Y_*^\nu(A,z) = {}^\circ_\circ Y_*^\nu(a_1, z + w_1) \cdots Y_*^\nu(a_k, z + w_k) {}^\circ_\circ f_{\bar{a}}(z, w_1, \ldots, w_k) c_{\bar{a}_1, \ldots, \bar{a}_k}, \tag{5.19}$$

where

$$c_{\bar{a}_1,\dots,\bar{a}_k} = \frac{\sigma(\bar{a})}{\sigma(\bar{a}_1)\cdots\sigma(\bar{a}_k)} \prod_{1\le i<j\le k} \frac{\varepsilon_2(\bar{a}_i,\bar{a}_j)}{\varepsilon_1(\bar{a}_i,\bar{a}_j)}. \tag{5.20}$$

(Lemma 4.3 is used here.) Similarly,

$$Y_*^\vee(B,z) = {}^\circ_\circ Y_*^\vee(b_1,z+x_1)\cdots Y_*^\vee(b_l,z+x_l){}^\circ_\circ f_{\bar{b}}(z,x_1,\dots,x_l)c_{\bar{b}_1,\dots,\bar{b}_l}, \tag{5.21}$$

where $c_{\bar{b}_1,\dots,\bar{b}_l}$ is defined as in (5.20). Thus

$$\begin{aligned}{}^\circ_\circ Y_*^\vee(A,z_1)Y_*^\vee(B,z_2){}^\circ_\circ &= f_{\bar{a}}(z_1,w_1,\dots,w_k)f_{\bar{b}}(z_2,x_1,\dots,x_l)c_{\bar{a}_1,\dots,\bar{a}_k}c_{\bar{b}_1,\dots,\bar{b}_l}\\ &\quad\times {}^\circ_\circ Y_*^\vee(a_1,z_1+w_1)\cdots Y_*^\vee(a_k,z_1+w_k)Y_*^\vee(b_1,z_2+x_1)\\ &\quad\times\cdots Y_*^\vee(b_l,z_2+x_l){}^\circ_\circ\\ &\in (\operatorname{End} V_L^T)\{z_1,z_2\}[[w_1,\dots,w_k,x_1,\dots,x_l]],\end{aligned} \tag{5.22}$$

and $\lim_{z_1^{1/p}\to z_2^{1/p}} {}^\circ_\circ Y_*^\vee(A,z_1)Y_*^\vee(B,z_2){}^\circ_\circ$ exists. That is, if $A(z_1,z_2)$ is the coefficient of any fixed monomial in $w_1,\dots,x_l$ in ${}^\circ_\circ Y_*^\vee(A,z_1)Y_*^\vee(B,z_2){}^\circ_\circ$, then $\lim_{z_1^{1/p}\to z_2^{1/p}} A(z_1,z_2)$ exists. From (4.54) we see that

$$Y_*^\vee(A,z_1)Y_*^\vee(B,z_2) = {}^\circ_\circ Y_*^\vee(A,z_1)Y_*^\vee(B,z_2){}^\circ_\circ f_{(\bar{a},\bar{b})}(z_1,w_1,\dots,w_k;z_2,x_1,\dots,x_l). \tag{5.23}$$

Fix a monomial

$$P = \prod_{1\le i\le k,\,1\le j\le l} w_i^{r_i}x_j^{s_j} \quad (r_i,s_j\ge 1) \tag{5.24}$$

in the w_i and x_j. We may and do choose $N\ge 0$ so large that the coefficients of P and of each monomial of lower total degree than P in

$$F_N = (z_1-z_2)^N f_{(\bar{a},\bar{b})}(z_1,w_1,\dots,w_k;z_2,x_1,\dots,x_l)F_{(\bar{a},\bar{b})}(z_1,z_2)$$

are polynomials in $z_1^{-1},z_2^{-1},z_1^{1/p}$ and $z_2^{1/p}$. In fact, the coefficient of P in F_N *is*

$$\begin{aligned}&\prod_{1\le i\le k}\prod_{1\le j\le l}\left(\frac{\partial}{\partial w_i}\right)^{r_i}\left(\frac{\partial}{\partial x_j}\right)^{s_j} F_N|_{w_i=x_j=0}\\ &= \sum C_{mnI}\prod_{r=0}^{p-1}(z_1^{1/p}-\omega_p^{-r}z_2^{1/p})^{i_r}z_1^{1/p-m}z_2^{1/p-n},\end{aligned}$$

where the sum is over $m,n,I=(i_0,\dots,i_{p-1})$ with $0\le i_r,\ m,n\le N$, and the C_{mnI} are constants. Then the coefficients of P and of each monomial of lower total degree than P in

$$(-z_2+z_1)^N f_{(\bar{b},\bar{a})}(z_2,x_1,\dots,x_l;z_1,w_1,\dots,w_k)F_{(\bar{b},\bar{a})}(z_2,z_1)$$

are also polynomials in z_i^{-1} and $z_i^{1/p}$ for $i=1,2$, and agree with the corresponding coefficients in F_N. Let $Y_p(z_1^{1/p},z_2^{1/p})$ denote the coefficient of P in

$$Y_*^\vee(,z_1)Y_*^\vee(B,z_2)(z_1-z_2)^N F_{(\bar{a},\bar{b})}(z_1,z_2) = {}^\circ_\circ Y_*^\vee(A,z_1)Y_*^\vee(B,z_2){}^\circ_\circ F_N. \tag{5.25}$$

By (5.22) and (5.23) we see that

$$Y_P(z_1^{1/p}, z_2^{1/p}) z^{-\bar{a}'_{(0)} - \langle \bar{a}'_{(0)}, \bar{a}'_{(0)} \rangle/2 + \langle \bar{a}', \bar{a}' \rangle/2} z_2^{-\bar{b}'_{(0)} - \langle \bar{b}'_{(0)}, \bar{b}'_{(0)} \rangle/2 + \langle \bar{b}', \bar{b}' \rangle/2}$$
$$\in (\operatorname{End} V_L^T)[[z_1^{1/p}, z_1^{-1/p}, z_2^{1/p}, z_2^{-1/p}]], \tag{5.26}$$

and $\lim_{z_1^{1/p} \to z_2^{1/p}} Y_P(z_1^{1/p}, z_2^{1/p})$ exists. The coefficient of P in

$$Y_*^\nu(A, z_1) Y_*^\nu(B, z_2)(z_1 - z_2)^N F_{(\bar{a}, \bar{b})}(z_1, z_2)$$

is

$$Y_P(z_1^{1/p}, z_2^{1/p})(z_1 - z_2)^{-(N-n)} = Y_P(z_1^{1/p}, z_2^{1/p}) z_1^{-(N-n)} \left(1 - \frac{z_2}{z_1}\right)^{-(N-n)}.$$

Similarly, reversing the roles of A and B and of z_1 and z_2 and noting that

$$\circ_\circ^\circ Y_*^\nu(B, z_2) Y_*^\nu(A, z_1) \circ_\circ^\circ = (c_\nu(\bar{a}, \bar{b}))^{-1} \circ_\circ^\circ Y_*^\nu(A, z_1) Y_*^\nu(B, z_2) \circ_\circ^\circ, \tag{5.27}$$

we find that

$$Y_*^\nu(B, z_2) Y_*^\nu(A, z_1) = c_\nu(\bar{a}, \bar{b})^{-1} \circ_\circ^\circ Y_*^\nu(A, z_1) Y_*^\nu(B, z_2) \circ_\circ^\circ$$
$$f_{(\bar{b}, \bar{a})}(z_2, x_1, \ldots, x_l; z_1, w_1, \ldots, w_k)$$

and that the coefficient of P in

$$c_\nu(\bar{a}, \bar{b}) Y_*^\nu(B, z_2) Y_*^\nu(A, z_1)(-z_2 + z_1)^N F_{(\bar{b}, \bar{a})}(z_2, z_1)$$

is also $Y_P(z_1^{1/p}, z_2^{1/p})$. Thus the coefficient of P in

$$c_\nu(\bar{a}, \bar{b}) Y_*^\nu(B, z_2) Y_*^\nu(A, z_1)(-z_2 + z_1)^N F_{(\bar{b}, \bar{a})}(z_2, z_1)$$

is

$$Y_P(z_1^{1/p}, z_2^{1/p})(-z_2 + z_1)^{-(N-n)} = Y_P(z_1^{1/p}, z_2^{1/p})(-z_2)^{-(N-n)} \left(1 - \frac{z_1}{z_2}\right)^{-(N-n)}.$$

Applying Proposition 5.1 with $A(z_1^{1/p}, z_2^{1/p})$ equal to $Y_P(z_1^{1/p}, z_2^{1/p})$, we see that the coefficient of P in

$$F_{(\bar{a}, \bar{b})}(z_1, z_2)(z_1 - z_2)^n Y_*^\nu(A, z_1) Y_*^\nu(B, z_2)$$
$$- c_\nu(\bar{a}, \bar{b}) F_{(\bar{b}, \bar{a})}(z_2, z_1)(-z_2 + z_1)^n Y_*^\nu(B, z_2) Y_*^\nu(A, z_1) \tag{5.28}$$

is

$$\operatorname{Res}_{z_0} z_2^{-1} z_0^{n-N} Y_P(z_1^{1/p}, z_2^{1/p}) e^{-z_0(\partial/\partial z_1)} \delta\left(\frac{z_1}{z_2}\right)$$
$$= \operatorname{Res}_{z_0} \frac{1}{p} z_2^{-1} z_0^{n-N} \sum_{r=0}^{p-1} Y_P(z_1^{1/p}, z_2^{1/p}) e^{-z_0(\partial/\partial z_1)} \delta\left(\omega_p^r \left(\frac{z_1}{z_2}\right)^{1/p}\right)$$

$$= \mathrm{Res}_{z_0} \frac{1}{p} z_2^{-1} z_0^{n-N} \sum_{r=0}^{p-1} \left(\left(\lim_{z_1^{1/p} \to \omega_p^{-r} z_2^{1/p}} Y_P((z_1+z_0)^{1/p}, z_2^{1/p}) \right) \right.$$
$$\times \omega_p^{r\sum_s \nu^s \bar{a}' + r\langle \sum_s \nu^s \bar{a}', \bar{a}' \rangle/2 - pr\langle \bar{a}', \bar{a}' \rangle/2}$$
$$\left. \times e^{-z_0(\partial/\partial z_1)} \left(\frac{z_1}{z_2}\right)^{\bar{a}'_{(0)} + \langle \bar{a}'_{(0)}, \bar{a}'_{(0)} \rangle/2 - \langle \bar{a}', \bar{a}' \rangle/2} \delta\left(\omega_p^r \left(\frac{z_1}{z_2}\right)^{1/p} \right) \right), \tag{5.29}$$

where s ranges over $\mathbb{Z}/p\mathbb{Z}$. Now we compute $\lim_{z_1^{1/p} \to \omega_p^{-r} z_2^{1/p}} Y_P((z_1+z_0)^{1/p}, z_2^{1/p})$. Note that $Y_P((z_1+z_0)^{1/p}, z_2^{1/p})$ is the coefficient of P in

$$\begin{aligned} &{}^{\circ}_{\circ} Y_*^\nu(A, z_1+z_0) Y_*^\nu(B, z_2) {}^{\circ}_{\circ} (z_1+z_0-z_2)^N F_{(\bar{a},\bar{b})}(z_1+z_0, z_2) \\ &\times f_{(\bar{a},\bar{b})}(z_1+z_0, w_1, \ldots, w_k; z_2, x_1, \ldots, x_l). \end{aligned}$$

From (4.51) we have

$$\begin{aligned} &\lim_{z_1^{1/p} \to \omega_p^{-r} z_2^{1/p}} {}^{\circ}_{\circ} Y_*^\nu(A, z_1+z_0) Y_*^\nu(B, z_2) {}^{\circ}_{\circ} \\ &= {}^{\circ}_{\circ} Y_*^\nu(\hat{\nu}^r A, z_2+z_0) Y_*^\nu(B, z_2) {}^{\circ}_{\circ} c_\nu(\bar{a} - \nu^r \bar{a}, \bar{b})(\hat{\nu}^r a^{-1}) \\ &\quad \times a \omega_p^{-r\sum \nu^s \bar{a}' - r\langle \sum \nu^s \bar{a}', \bar{a}' \rangle/2 + rp\langle \bar{a}', \bar{a}' \rangle/2}. \end{aligned}$$

We also have

$$\begin{aligned} &\lim_{z_1^{1/p} \to \omega_p^{-r} z_2^{1/p}} F_{(\bar{a},\bar{b})}(z_1+z_0, z_2) f_{(\bar{a},\bar{b})}(z_1+z_0, w_1, \ldots, w_k; z_2, x_1, \ldots, x_l) \\ &= F_{(\nu^r \bar{a},\bar{b})}(z_2+z_0, z_2) f_{(\nu^r \bar{a},\bar{b})}(z_2+z_0, w_1, \ldots, w_k; z_2, x_1, \ldots, x_l). \end{aligned}$$

Thus $\lim_{z_1^{1/p} \to \omega_p^{-r} z_2^{1/p}} Y_P((z_1+z_0)^{1/p}, z_2^{1/p})$ is the coefficient of P in

$$\begin{aligned} &{}^{\circ}_{\circ} Y_*^\nu(\hat{\nu}^r A, z_2+z_0) Y_*^\nu(B, z_2) {}^{\circ}_{\circ} c_\nu(\bar{a} - \nu^r \bar{a}, \bar{b})(\hat{\nu}^r a^{-1}) a z_0^N F_{(\nu^r \bar{a},\bar{b})}(z_2+z_0, z_2) \\ &\times f_{(\nu^r \bar{a},\bar{b})}(z_2+z_0, w_1, \ldots, w_k; z_2, x_1, \ldots, x_l) \omega^{-r\sum \nu^s \bar{a}' - r\langle \sum \nu^s \bar{a}', \bar{a}' \rangle/2 + rp\langle \bar{a}', \bar{a}' \rangle/2}. \end{aligned} \tag{5.30}$$

We conclude that the coefficient of P in (5.28) is the coefficient of P in

$$\begin{aligned} &\mathrm{Res}_{z_0} \frac{1}{p} z_2^{-1} z_0^n \sum_{r=0}^{p-1} \left({}^{\circ}_{\circ} Y_*^\nu(\hat{\nu}^r A, z_2+z_0) Y_*^\nu(B, z_2) {}^{\circ}_{\circ} \delta\left(\omega_p^r \left(\frac{z_1 - z_0}{z_2}\right)^{1/p} \right) \right. \\ &\times F_{(\nu^r \bar{a},\bar{b})}(z_2+z_0, z_2) f_{(\nu^r \bar{a},\bar{b})}(z_2+z_0, w_1, \ldots, w_k; z_2, x_1, \ldots, x_l) \\ &\left. \times c_\nu(\bar{a} - \nu^r \bar{a}, \bar{b})(\hat{\nu}^r a^{-1}) a \left(\frac{z_1 - z_0}{z_2}\right)^{\bar{a}'_{(0)} + \langle \bar{a}'_{(0)}, \bar{a}'_{(0)} \rangle/2 - \langle \bar{a}', \bar{a}' \rangle/2} \right). \end{aligned} \tag{5.31}$$

Therefore the operators in (5.28) and in (5.31) are the same. This last assertion is independent of P and N.

On the other hand, just as in the proof of Theorem 5.1 of [7], we obtain

$$\begin{aligned} Y_*(\hat{\nu}^r A, z_0) B = {}^{\circ}_{\circ} &Y_*(\hat{\nu}^r a_1, z_0+w_1) \cdots Y_*(\hat{\nu}^r a_k, z_0+w_k) Y_*(b_1, x_1) \cdots Y_*(b_l, x_l) {}^{\circ}_{\circ} \iota(1) \\ &\times \prod_{1 \le i \le k, 1 \le j \le l} (z_0 + w_i - x_j)^{\langle \hat{\nu}^r \bar{a}_i, \bar{b}_j \rangle}. \end{aligned}$$

By (5.17) and (5.19) we have

$$\begin{aligned}Y_*^\nu(Y_*(\hat{\nu}^r A, z_0)B, z_2) &= {}^{\circ}_{\circ} Y_*^\nu(\hat{\nu}^r a_1, z_2 + z_0 + w_1) \cdots Y_*^\nu(\hat{\nu}^r a_k, z_2 + z_0 + w_k) \\ &\quad \times Y_*^\nu(b_1, z_2 + x_1) \cdots Y_*^\nu(b_l, z_2 + x_l) {}^{\circ}_{\circ} f_{\bar{a}}(z_2 + z_0, w_1, \ldots, w_k) \\ &\quad \times f_{\bar{b}}(z_2, x_1, \ldots, x_l) f_{(\nu^r\bar{a}, \bar{b})}(z_2 + z_0, w_1, \ldots, w_k; z_2, x_1, \ldots, x_l) \\ &\quad \times \prod_{1 \le i \le k, 1 \le j \le l} \prod_{s=0}^{p-1} ((z_2 + z_0 + w_i)^{1/p} \\ &\quad - \omega_p^s (z_2 + x_j)^{1/p})^{-\langle \nu^r \bar{a}_i', \bar{b}_j' \rangle} \\ &\quad \times \prod_{1 \le i \le k, 1 \le j \le l} (z_0 + w_i - x_j)^{\langle \hat{\nu}^r \bar{a}_i', \bar{b}_j' \rangle} \\ &\quad \times c_{\nu^r \bar{a}_1, \ldots, \nu^r \bar{a}_k} c_{\bar{b}_1, \ldots, \bar{b}_l} (\tau(\nu^r \bar{a}, \bar{b}))^{-1}, \end{aligned} \tag{5.32}$$

where we have used the relation

$$c_{\nu^r \bar{a}_1, \ldots, \nu^r \bar{a}_k, \bar{b}_1, \ldots, \bar{b}_l} = c_{\nu^r \bar{a}_1, \ldots, \nu^r \bar{a}_k} c_{\bar{b}_1, \ldots, \bar{b}_l} (\tau(\nu^r \bar{a}, \bar{b}))^{-1}$$

(see (5.7) and (5.20)). We next prove that

$$\begin{aligned}&\prod_{1 \le i \le k, 1 \le j \le l} \prod_{s=0}^{p-1} ((z_2 + z_0 + w_i)^{1/p} - \omega_p^s (z_2 + x_j)^{1/p})^{-\langle \nu^r \bar{a}_i', \bar{b}_j' \rangle} \\ &\quad \times \prod_{1 \le i \le k, 1 \le j \le l} (z_0 + w_i - x_j)^{\langle \nu^r \bar{a}_i', \bar{b}_j' \rangle} \\ &= z_0^{\langle \nu^r \bar{a}', \bar{b}' \rangle} \prod_{s=0}^{p-1} ((z_2 + z_0)^{1/p} - \omega_p^s z_2^{1/p})^{-\langle \nu^r \bar{a}', \bar{b}' \rangle}.\end{aligned}$$

For this purpose we introduce

$$g(z_2, z_0, w, x) = \prod_{s=0}^{p-1} ((z_2 + z_0 + w)^{1/p} - \omega_p^s (z_2 + x)^{1/p})^c (z_0 + w - x)^{-c}, \tag{5.33}$$

where $c \in \mathbb{C}$, w, x are formal variables, and the formal series is interpreted as above. It suffices to prove that $g(z_2, z_0, w, x) = g(z_2, z_0, 0, 0)$, or that $(\partial g/\partial w) = (\partial g/\partial x) = 0$. Now we calculate $(\partial g/\partial w)$:

$$\begin{aligned}\frac{\partial g}{\partial w} &= \sum_{s=0}^{p-1} \frac{c}{p} (z_2 + z_0 + w)^{1/p - 1} ((z_2 + z_0 + w)^{1/p} \\ &\quad - \omega_p^s (z_2 + x)^{1/p})^{-1} g - c(z_0 + w - x)^{-1} g,\end{aligned}$$

which is zero because

$$\sum_{s=0}^{p-1} \frac{1}{p} (z_2 + z_0 + w)^{1/p - 1} ((z_2 + z_0 + w)^{1/p} - \omega_p^s (z_2 + x)^{1/p})^{-1} = (z_0 + w - x)^{-1}.$$

Similarly, $(\partial g/\partial x) = 0$.

Finally, we see from (5.28) that the operator $Y_*^\nu(Y_*(\hat{\nu}^r A, z_0)B, z_2)$ is equal to

$$\begin{aligned}&{}^\circ_\circ Y_*^\nu(\hat{\nu}A, z_2+z_0)Y_*^\nu(B,z_2){}^\circ_\circ f_{(\nu^r\bar{a},\bar{b})}(z_2+z_0, w_1, \ldots, w_k; z_2, x_1, \ldots, x_l)\\ &\times(\tau(\nu^r\bar{a},\bar{b}))^{-1} z_0^{\langle \nu^r\bar{a}',\bar{b}'\rangle}\prod_{s=0}^{p-1}((z_2+z_0)^{1/p}-\omega_p^s z_2^{1/p})^{-\langle \nu^r\bar{a}',\bar{b}'\rangle}.\end{aligned} \tag{5.34}$$

Combining (5.28)–(5.31) and (5.34) completes the proof. □

Remark 5.3. In the case in which L is a direct sum of several copies of the root lattice of the Lie algebra $\mathrm{sl}(2,\mathbb{C})$ and $\nu = -1$, Theorem 5.2 was obtained in [16], and both the twisted Z-algebra relations [24–28] and the twisted parafermion algebra relations [33] associated with the twisted vertex operator constructions of $A_1^{(1)}$ were recovered as a consequence of this Jacobi identity and a multi-operator extension of it.

The Jacobi identity for relative untwisted vertex operators [6, 7, 10] is a special case of Theorem 5.2:

Corollary 5.4. *Let $u, v \in V_L$. Then*

$$\begin{aligned}&z_0^{-1}\left(\frac{z_1-z_2}{z_0}\right)^{-\langle \bar{a}',\bar{b}'\rangle}\delta\left(\frac{z_1-z_2}{z_0}\right)Y_*(u,z_1)Y_*(v,z_2)\\ &\quad - c(\bar{a},\bar{b})z_0^{-1}\left(\frac{z_2-z_1}{z_0}\right)^{-\langle \bar{a}',\bar{b}'\rangle}\delta\left(\frac{z_2-z_1}{-z_0}\right)Y_*(v,z_2)Y_*(u,z_2)\\ &= z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)Y_*(Y_*(u,z_0)v,z_2)\left(\frac{z_1-z_0}{z_2}\right)^{\bar{a}'}.\end{aligned} \tag{5.35}$$

Proof. If $\nu = 1$, $p = 1$, $c_0^\nu(\cdot,\cdot) = c_0(\cdot,\cdot)$ and $T = \mathbb{C}\{L\}$, then $V_L^T = V_L$ and $Y_*^\nu(u,z) = Y_*(u,z)$. Moreover, $F_{(\alpha,\beta)} = (z_1 - z_2)^{-\langle \alpha',\beta'\rangle}$, $G_{(\alpha,\beta)} = 1$ and $\tau(\alpha,\beta) = 1$ for $\alpha, \beta \in L$ in this case,. Now the Jacobi identity (5.35) follows immediately from (5.11). □

We also include the Jacobi identity for ordinary (unrelativized) twisted vertex operators (and therefore the commutator formula for these operators ([13, 19]) as follows:

Corollary 5.5. *In the settings of Remarks* 2.2, 4.2 *and* 4.5, *for $u, v \in V_L$, we have $Y_*^\nu(u,z) = Y_\nu(u,z)$ and*

$$\begin{aligned}&z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)Y_\nu(u,z_1)Y_\nu(v,z_2) - z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right)Y_\nu(v,z_2)Y_\nu(u,z_1)\\ &= \frac{1}{p}z_2^{-1}\sum_{r=0}^{p-1}\delta\left(\omega_p^r\left(\frac{z_1-z_0}{z_2}\right)^{1/p}\right)Y_\nu(Y(\hat{\nu}^r u,z_0)v,z_2).\end{aligned} \tag{5.36}$$

Proof. From (2.10) and (4.23),

$$c_\nu(\bar{a} - \nu^r\bar{a}, \bar{b}) = \prod_{s=0}^{p-1} (-\omega_p^s)^{\langle \nu^s(\bar{a}' - \nu^r\bar{a}), \bar{b}\rangle} = \omega_p^{r\langle \sum \nu^s\bar{a}, \bar{b}\rangle},$$

$$(\hat{\nu}^r a^{-1})a = \omega_p^{r\sum \nu^s\bar{a} + r\langle \sum \nu^s\bar{a}, \bar{a}\rangle/2}.$$

Since

$$\left\langle \sum_s \nu^s\bar{a}, \bar{a} \right\rangle \in 2\mathbb{Z} \quad \text{for } \alpha \in L$$

by (2.8) we see that

$$\langle \bar{a}_{(0)}, \bar{a}_{(0)} \rangle \in \frac{2}{p}\mathbb{Z}$$

and that

$$\left(\frac{z_1 - z_0}{z_2}\right)^{\bar{a}_{(0)}} \in (\operatorname{End} V_L^T)[[z_1^{1/p}, z_1^{-1/p}, z_2^{1/p}, z_2^{-1/p}, z_0]].$$

Using the basic property of the δ-function we have

$$\left(\frac{z_1 - z_0}{z_2}\right)^{\bar{a}_{(0)} + \langle \bar{a}_{(0)}, \bar{a}_{(0)}\rangle/2 - \langle \bar{a}, \bar{a}\rangle/2} \delta\left(\omega_p^r \left(\frac{z_1 - z_0}{z_2}\right)^{1/p}\right)$$

$$= \omega_p^{-r\sum \nu^s\bar{a} - r\langle \sum \nu^s\bar{a}, \bar{a}\rangle/2} \delta\left(\omega_p^r \left(\frac{z_1 - z_0}{z_2}\right)^{1/p}\right).$$

Since L is even we see that

$$G_{(\nu^r\bar{a}, \bar{b})}(z_2 + z_0, z_2) z_0^{-\langle \nu^r\bar{a}, \bar{b}\rangle} = F_{(\bar{a}, \bar{b})}(z_2 + z_0, z_2).$$

Then the Jacobi identity (5.11) reduces to

$$z_0^{-1} F_{(\bar{a}, \bar{b})}(z_1, z_2) \delta\left(\frac{z_1 - z_2}{z_0}\right) Y_\nu(u, z_1) Y_\nu(v, z_2)$$

$$- c_\nu(\bar{a}, \bar{b}) z_0^{-1} F_{(\bar{b}, \bar{a})}(z_2, z_1) \delta\left(\frac{z_2 - z_1}{-z_0}\right) Y_\nu(v, z_2) Y_\nu(u, z_1)$$

$$= \frac{1}{p} z_2^{-1} \sum_{r=0}^{p-1} F_{(\nu^r\bar{a}, \bar{b})}(z_2 + z_0, z_2) Y_\nu(Y(\hat{\nu}^r u, z_0)v, z_2) \omega_p^{r\langle \sum \nu^r\bar{a}, \bar{b}\rangle}$$

$$\times \delta\left(\omega_p^r \left(\frac{z_1 - z_0}{z_2}\right)^{1/p}\right) \tag{5.37}$$

(recall that $\tau(\alpha, \beta) = 1$ for $\alpha, \beta \in L$ in this case). Noting that

$$\prod_{s=0}^{p-1} (z_2^{1/p} - \omega_p^{-s} z_1^{1/p})^{-\langle \nu^s\bar{b}, \bar{a}\rangle} = (c_\nu(a, \bar{b}))^{-1} \prod_{s=0}^{p-1} (-\omega_p^{-s} z_2^{1/p} + z_2^{1/p})^{-\langle \nu^s\bar{a}, \bar{b}\rangle}$$

and that at least one of the two binomials $(z_1^{1/p} - \omega_p^s z_2^{1/p})^{\langle v^s\bar{a},\bar{b}\rangle}$ and $(-\omega_p^s z_2^{1/p} + z_1^{1/p})^{-\langle v^s\bar{a},\bar{b}\rangle}$ is a polynomial in $z_1^{1/p}$ and $z_2^{1/p}$, and multiplying (5.37) by $\prod_{s=0}^{p-1}(z_1^{1/p} - \omega_p^{-s} z_2^{1/p})^{\langle v^s\bar{a},\bar{b}\rangle}$, we have

$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right)Y_v(u,z_1)Y_v(v,z_2) - z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right)Y_v(v,z_2)Y_v(u,z_1)$$

$$= \frac{1}{p}z_2^{-1}\sum_{r=0}^{p-1} F_{(v^r\bar{a},\bar{b})}(z_2 + z_0, z_2)\prod_s (z_1^{1/p} - \omega_p^{-s} z_2^{1/p})^{\langle v^r\bar{a},\bar{b}\rangle}$$

$$\times Y_v(Y(\hat{v}^r u, z_0)v, z_2)\omega_p^{r\langle \sum v^s\bar{a},\bar{b}\rangle}\delta\left(\omega_p^r\left(\frac{z_1 - z_0}{z_2}\right)^{1/p}\right).$$

Now (5.36) follows from

$$\prod_{s=0}^{p-1}(z_1^{1/p} - \omega_p^{-s} z_2^{1/p})^{\langle v^r\bar{a},\bar{b}\rangle}\delta\left(\omega_p^r\left(\frac{z_1 - z_0}{z_2}\right)^{1/p}\right)$$

$$= (F_{(v^r\bar{a},\bar{b})}(z_2 + z_0, z_2))^{-1}\omega^{-r\langle \sum v^s\bar{a},\bar{b}\rangle}\delta\left(\omega_p^r\left(\frac{z_1 - z_0}{z_2}\right)^{1/p}\right). \quad \square$$

The following corollary is used to construct the "moonshine modules" based on order p isometries of the Leech lattice for the odd primes 3, 5, 7 and 13 [8]:

Corollary 5.6. *Let p be an odd prime and let v be an isometry of L having no nonzero fixed points. Let L_0 be an even sublattice of L such that* (2.9), (2.10) *and* (2.13) *in Remark* 2.2 *hold for the restrictions of $c_0(\cdot,\cdot)$, $c_0^v(\cdot,\cdot)$, and $\varepsilon_0(\cdot,\cdot)$ to L_0. Take $\boldsymbol{h}_* = 0$.* Then for *$u, v \in V_{L_0}$, we have $Y_*^v(u,z) = Y_v(u,z)$ and*

$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right)Y_v(u,z_1)Y_v(v,z_2) - z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right)Y_v(v,z_2)Y_v(u,z_1)$$

$$= \frac{1}{p}z_2^{-1}\sum_{r=0}^{p-1}\delta\left(\omega_p^r\left(\frac{z_1 - z_0}{z_2}\right)^{1/p}\right)Y_v(Y(\hat{v}^r u, z_0)v, z_2)(\hat{v}^r a^{-1})a \tag{5.38}$$

as operators on V_L^T.

Proof. Note that $\boldsymbol{h}_{(0)} = 0$ and $\alpha_{(0)} = 0$ for $\alpha \in \boldsymbol{h}$ because v is fixed-point free. Now the proof is almost the same as that of Corollary 5.5, except that we do not replace $(\hat{v}^r a^{-1})a$ by $\omega_p^{r\sum v^s\bar{a} + r\langle \sum v^s\bar{a},\bar{a}\rangle/2}$. $\square$

6. The Virasoro algebras

In this section we study the representations of the Virasoro algebra, with basis $\{L_n \mid n \in \mathbb{Z}\} \cup \{c\}$ and with the usual commutation relations

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{1}{12}(m^3 -)\delta_{m+n,0}c \quad \text{for } m, n \in \mathbb{Z}, \tag{6.1}$$

on both V_L and V_L^T, in terms of the relative (untwisted and twisted) vertex operators associated with a canonical quadratic element of weight 2 in V_L. The natural operator Δ_z (4.42) incorporated into the definition of the relative twisted vertex operators plays a fundamental role. We use the action of L_0 to reinterpret the weight gradations of V_L and V_L^T. The reader can refer to [7, 14] for similar discussions for relative untwisted vertex operators and background.

Recall that $\{\beta_1, \ldots, \beta_d\}$ is an orthonormal basis of $\boldsymbol{h}_*^{\perp}$. Set

$$\omega = \frac{1}{2}\sum_{i=1}^{d} \beta_i(-1)^2 \tag{6.2}$$

and set

$$L(z) = Y_*(\omega, z) = \sum_{n\in\mathbb{Z}} L(n)z^{-n-2}, \tag{6.3}$$

i.e.,

$$L(n) = \omega_{n+1} \quad \text{for } n \in \mathbb{Z} \tag{6.4}$$

(recall (3.41), the definition of the components of a relative untwisted vertex operator). We also define operators $L^\nu(n)$ on V_L^T as the following generating function coefficients:

$$L^\nu(z) = Y_*^\nu(\omega, z) = \sum_{n\in\mathbb{Z}} L^\nu(n)z^{-n-2}, \tag{6.5}$$

i.e.,

$$L^\nu(n) = \omega_{n+1}^\nu \quad \text{for } n \in \mathbb{Z} \tag{6.6}$$

(see (4.46)).

We have the important relation between $L(-1)$ and differentiation:

Proposition 6.1. *For all $v \in V_L$,*

$$Y_*^\nu(L(-1)v, z) = \frac{\mathrm{d}}{\mathrm{d}z} Y_*^\nu(v, z). \tag{6.7}$$

Proof. First we observe that a special case of (6.7), for $v = \iota(a)$, follows from (4.55) and the fact that

$$L(-1)\iota(a) = \bar{a}'(-1)\iota(a).$$

It is easy to see from the definition (3.40) that for all $v \in V_L$,

$$Y_*(L(-1)v, z) = \frac{\mathrm{d}}{\mathrm{d}z} Y_*(v, z).$$

Thus we have

$$Y_*(\mathrm{e}^{z_0 L(-1)}v, z) = \mathrm{e}^{z_0(\mathrm{d}/\mathrm{d}z)} Y_*(v, z) = Y_*(v, z + z_0).$$

We can now apply both sides to $\iota(1)$ and invoke (3.45) to obtain

$$e^{z_0 L(-1)}v = Y_*(v, z_0)\cdot\iota(1).$$

On the other hand, as operators on V_L^T,

$$Y_*^\vee(Y_*(v, z_0)\cdot\iota(1), z) = Y_*^\vee(v, z + z_0)$$

by the proof of Theorem 5.2. Hence

$$e^{z_0(d/dz)}Y_*^\vee(v,z) = Y_*^\vee(v, z + z_0) = Y_*^\vee(e^{z_0 L(-1)}v, z),$$

and (6.7) follows by extracting the coefficient of z_0. □

Combining the Jacobi identity with Proposition 6.1 we have

$$\begin{aligned}
[L^\vee(z_1), Y_*^\vee(v, z_2)] &= \mathrm{Res}_{z_0} z_2^{-1} Y_*^\vee(L(z_0)v, z_2) e^{-z_0(\partial/\partial z_1)}\delta(z_1/z_2) \\
&= z_2^{-1}\left(\frac{d}{dz} Y_*^\vee(v, z_2)\right)\delta(z_1/z_2) - z_2^{-1} Y_*^\vee(L(0)v, z_2)\frac{\partial}{\partial z_1}\delta(z_1/z_2) \\
&\quad + z_2^{-1}\mathrm{Res}_{z_0}\sum_{n>0} Y_*^\vee(L(n)v, z_2) z_0^{-n-2} e^{-z_0(\partial/\partial z_1)}\delta(z_1/z_2)
\end{aligned} \tag{6.8}$$

for all $v \in V_L$. Equating the coefficient of z_1^{-1} and changing z_2 to z, we get

Proposition 6.2. *For all $v \in V_L$,*

$$[L^\vee(-1), Y_*^\vee(v, z)] = \frac{d}{dz} Y_*^\vee(v, z) = Y_*^\vee(L(-1)v, z). \tag{6.9}$$

We call a nonzero vector $v \in V_L$ (resp. $v \in V_L^T$) a *weight vector* if v satisfies the following condition:

$$L(0)v = hv \text{ (resp., } L^\vee(0)v = hv) \quad \text{for some } h \in \mathbb{C} \tag{6.10}$$

and we call h the weight of v. If v further satisfies the condition

$$L(n)v = 0 \text{ (resp., } L^\vee(n)v = 0) \quad \text{for } n > 0, \tag{6.11}$$

we call v a *lowest weight vector.*

By (6.8) we have

Proposition 6.3. *If $v \in V_L$ is a lowest weight vector with the weight h, then*

$$[L^\vee(z_1), Y_*^\vee(v, z_2)] = z_2^{-1}\left(\frac{d}{dz_2} Y_*^\vee(v, z_2)\right)\delta(z_1/z_2) - h z_2^{-1} Y_*^\vee(v, z_2)\frac{\partial}{\partial z_1}\delta(z_1/z_2), \tag{6.12}$$

or equivalently,

$$[L^{\nu}(m), Y_*^{\nu}(v,z)] = \left(z^{n+1}\frac{\mathrm{d}}{\mathrm{d}z} + h(n+1)z^n\right) Y_*^{\nu}(v,z) \tag{6.13}$$

for $m \in \mathbb{Z}$.

Note that any $a \in \hat{L}$ is a lowest weight vector with weight $\frac{1}{2}\langle \bar{a}', \bar{a}'\rangle$. Thus by (6.12),

$$[L^{\nu}(z_1), Y_*^{\nu}(a,z_2)] = z_2^{-1}\left(\frac{\mathrm{d}}{\mathrm{d}z_2} Y^{\nu}(a,z_2)\right)\delta(z_1/z_2)$$
$$-\frac{1}{2}\langle \bar{a}', \bar{a}'\rangle z_2^{-1} Y_*^{\nu}(a,z_2)\frac{\partial}{\partial z_1}\delta(z_1/z_2). \tag{6.14}$$

Applying (6.13) to $v = \alpha(-1)$, which is a lowest weight vector with weight $h = 1$ for $\alpha \in \boldsymbol{h}_*^{\perp}$, and equating the coefficients of z^{-n-1}, we obtain

$$[L^{\nu}(m), \alpha(n)] = -n\alpha(m+n) \tag{6.15}$$

for $n \in (1/p)\mathbb{Z}$,

Now in (6.8), taking $v = \omega$ and noting that

$$L(n)\omega = 0, \quad \text{if } n > 0 \text{ and } n \neq 2, \qquad L(0)\omega = 2\omega, \tag{6.16}$$

$$L(2)\omega = \frac{1}{2}\dim \boldsymbol{h}_*^{\perp},$$

we have

$$[L^{\nu}(z_1), L^{\nu}(z_2)] = z_2^{-1}\left(\frac{\mathrm{d}}{\mathrm{d}z_2} L^{\nu}(z_2)\right)\delta(z_1/z_2) - 2z_2^{-1}L^{\nu}(z_2)\frac{\partial}{\partial z_1}\delta(z_1/z_2)$$
$$-\frac{1}{12}(\dim \boldsymbol{h}_*^{\perp})z_2^{-1}\left(\frac{\partial}{\partial z_1}\right)^3\delta(z_1/z_2). \tag{6.17}$$

Equating the coefficients of $z_1^{-m-2}z_2^{-n-2}$, we obtain

$$[L^{\nu}(m), L^{\nu}(n)] = (m-n)L^{\nu}(m+n) + \frac{1}{12}(m^3 - m)\dim \boldsymbol{h}_*^{\perp}\delta_{m+n,0} \tag{6.18}$$

for $m, n \in \mathbb{Z}$. We conclude.

Proposition 6.4. *The operators $L^{\nu}(n)$, $n \in \mathbb{Z}$ and I span a copy of the Virasoro algebra, and the operators $L^{\nu}(n)$ provide a representation of the Virasoro algebra on V_L^T with*

$$L_n \mapsto L^{\nu}(n) \quad \text{for } n \in \mathbb{Z} \qquad c \mapsto \dim \boldsymbol{h}_*^{\perp}. \tag{6.19}$$

Finally, we justify the weight gradation of V_L^T introduced in Section 4 by using the definition of weight given by (6.10). For this purpose we need to study $L^{\nu}(z)$ in more detail. It is easy to see from (4.42) that

$$e^{\Delta_z}\alpha(-1)\beta(-1) = \alpha(-1)\beta(-1) + \left(2c_{110} + \sum_{i=1}^{p-1} c_{11i}\omega_p^{ki} + \omega_p^{-ki}\right)\langle\alpha,\beta\rangle \tag{6.20}$$

for $k \in \{0, \dots, p-1\}$ and $\alpha \in (\boldsymbol{h}_*^{\perp})_{(k)}, \beta \in (\boldsymbol{h}_*^{\perp})_{(-k)}$. The numbers c_{11r} for $r \in 0, \dots, p-1$ are defined in (4.41) and are given by

$$c_{110} = -\frac{1}{2p^2}\sum_{i=1}^{p-1}\frac{\omega_p^i}{(1-\omega_p^i)^2} \qquad c_{11r} = \frac{1}{2p^2}\frac{\omega_p^{-r}}{(1-\omega_p^{-r})^2} \quad \text{for } r \neq 0.$$

Define constants

$$c_k = \sum_{r=1}^{p-1}\frac{\omega_p^{kr}}{(1-\omega_p^r)^2}. \tag{6.21}$$

Then

$$c_0 = -(p-1)(p-5)/12, \qquad c_1 = -(p^2-1)/12. \tag{6.22}$$

Using (6.22) and the recursive formula

$$c_k = c_{k-1} + \frac{p+1}{2} - k + 1, \tag{6.23}$$

we find that

$$c_k = (k-1)(p+1-k)/2 - (p^2-1)/12. \tag{6.24}$$

Therefore,

$$2c_{110} + \sum_{i=1}^{p-1} c_{11i}(\omega_p^{ki} + \omega_p^{-ki}) = \frac{c_{k+1} + c_{p-k+1} - 2c_1}{2p^2} = \frac{k(p-k)}{2p^2}. \tag{6.25}$$

By (6.20) and (6.25) we have an explicit expression for the vertex operator $Y_*^{\nu}(\alpha(-1)\beta(-1),z)$:

$$Y_*^{\nu}(\alpha(-1)\beta(-1),z) = {}^{\circ}_{\circ}\alpha(z)\beta(z){}^{\circ}_{\circ} + \frac{k(p-k)}{2p^2}\langle\alpha,\beta\rangle. \tag{6.26}$$

Recall the canonical quadratic element ω (6.2). Then by the last formula

$$L^{\nu}(z) = \frac{1}{2}\sum_{i=1}^{d} {}^{\circ}_{\circ}\beta_i(z)\beta_i(z){}^{\circ}_{\circ} + \frac{1}{4p^2}\sum_{k=1}^{p-1} k(p-k)\dim(\boldsymbol{h}_*^{\perp})_{(k)}. \tag{6.27}$$

It is obvious now that

$$L^{\nu}(0)1 = \frac{1}{4p^2}\sum_{k=1}^{p-1} k(p-k)\dim(\boldsymbol{h}_*^{\perp})_{(k)}1 \tag{6.28}$$

for $1 \in S[\nu]$ (cf. (4.9)), and therefore the weight defined in (4.10) is exactly the weight defined by the $L^{\nu}(0)$-eigenvalue (6.10). Similarly, the weight of $t = 1 \otimes t \in V_L^T$ for $t \in T_\alpha$ ($\alpha \in \boldsymbol{h}_{(0)}$) is $\frac{1}{2}\langle\alpha',\alpha'\rangle + (1/4p^2)\sum_{k=1}^{p-1} k(p-k)\dim(\boldsymbol{h}_*^{\perp})_{(k)}$.

Remark 6.5. The weight of 1 in (6.28) is closely related to the second Bernoulli polynomial, defined by $B_2(x) = x^2 - x + 1/6$. In fact, one can check that wt 1 is equal to $-\frac{1}{4}\sum_{k=0}^{p-1} B_2(k/p)\dim(\boldsymbol{h}_*^{\perp})_{(k)} + \dim \boldsymbol{h}_*^{\perp}/24$ (cf. [29,9]).

7. Twisted modules for V_L

In this section we recall certain notions of vertex algebra and of twisted module. We also present a family of $\hat{\nu}$-twisted modules V_L^T for the vertex algebra V_L in the case in which L is even.

First we recall the definition of vertex algebra used in [7]. (This is different from the original definition in [2].) A *vertex algebra* is a $\mathbb{Z}$-graded vector space

$$V = \coprod_{n\in\mathbb{Z}} V_n; \qquad \text{for } n \in V_n,\ n = \operatorname{wt} v; \tag{7.1}$$

equipped with a linear map

$$V \to (\operatorname{End} V)[[z,z^{-1}]] \qquad v \mapsto Y(v,z) = \sum_{n\in\mathbb{Z}} v_n z^{-n-1} \quad (v_n \in \operatorname{End} V) \tag{7.2}$$

and with two distinguished vectors $\mathbf{1} \in V_0, \omega \in V_2$ satisfying the following conditions for $u, v \in V$:

$$u_n v = 0 \quad \text{for } n \text{ sufficiently large}; \tag{7.3}$$

$$Y(\mathbf{1},z) = 1; \tag{7.4}$$

$$Y(v,z)\mathbf{1} \in V[[z]] \quad \text{and} \quad \lim_{z\to 0} Y(v,z)\mathbf{1} = v; \tag{7.5}$$

$$z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)Y(u,z_1)Y(v,z_2) - z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right)Y(v,z_2)Y(u,z_1)$$

$$= z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)Y(Y(u,z_0)v,z_2); \tag{7.6}$$

$$[L(m),L(n)] = (m-n)L(m+n) + \frac{1}{12}(m^3-m)\delta_{m+n,0}(\operatorname{rank} V) \tag{7.7}$$

for $m, n \in \mathbb{Z}$, where

$$L(n) = \omega_{n+1} \quad \text{for } n \in \mathbb{Z}, \text{ i.e., } Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2} \tag{7.8}$$

and

$$\operatorname{rank} V \in \mathbb{Q}; \tag{7.9}$$

$$L(0)v = nv = (\operatorname{wt} v)v \quad \text{for } v \in V_n \ (n \in \mathbb{Z}); \tag{7.10}$$

$$\frac{\mathrm{d}}{\mathrm{d}z} Y(v, z) = Y(L(-1)v, z). \tag{7.11}$$

This completes the definition. We denote the vertex algebra just defined by $(V, Y, \mathbf{1}, \omega)$ (or briefly, by V). The series $Y(v, z)$ are called *vertex operators.*

An *automorphism go* of the vertex algebra V is a linear automorphism of V preserving $\mathbf{1}$ and ω such that the actions of g and $Y(v, z)$ on V are compatible in the sense that

$$gY(v, z)g^{-1} = Y(gv, z) \tag{7.12}$$

for $v \in V$. Then $gV_n \subset V_n$ for $n \in \mathbb{Z}$ and V is a direct sum of the eigenspaces of g:

$$V = \coprod_{j \in \mathbb{Z}/N\mathbb{Z}} V^j, \tag{7.13}$$

where N is the order or g and $V^j = \{v \in V \mid v = \omega_N^j v\}$.

We next recall the notion of g-twisted module (see [10, 5]; this notion records the properties obtained in [19], Section 3.3 of [13], and [20]). Let $(V, Y, \mathbf{1}, \omega)$ be a vertex algebra and let g be an automorphism of V of order N. A g-twisted *module* M for $(Y, V, \mathbf{1}, \omega)$ is a $\mathbb{Q}$-graded vector space

$$M = \coprod_{n \in \mathbb{Q}} M_n; \quad \text{for } w \in M_n,\ n = \operatorname{wt} w; \tag{7.14}$$

equipped with a linear map

$$\begin{aligned} &V \to (\operatorname{End} M)[[z^{1/N}, z^{-1/N}]] \\ &v \mapsto Y(v, z) = \sum_{n \in (1/N)\mathbb{Z}} v_n z^{-n-1} \quad (v_n \in \operatorname{End} M) \end{aligned} \tag{7.15}$$

satisfying the following conditions for $u, v \in V$ and $w \in M$ and $j \in \mathbb{Z}$:

$$Y(v, z) = \sum_{n \in j/N + \mathbb{Z}} v_n z^{-n-1} \quad \text{for } v \in V^j; \tag{7.16}$$

$$u_n w = 0 \quad \text{for } n \text{ sufficiently large}; \tag{7.17}$$

$$Y(\mathbf{1}, z) = 1; \tag{7.18}$$

$$\begin{aligned} &z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y(u, z_1) Y(v, z_2) - z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right) Y(v, z_2) Y(u, z_1) \\ &= z_2^{-1} \frac{1}{N} \sum_{j \in \mathbb{Z}/N\mathbb{Z}} \delta\left(\omega_N^j \frac{(z_1 - z_0)^{1/N}}{z_2^{1/N}}\right) Y(Y(g^j u, z_0)v, z_2), \end{aligned} \tag{7.19}$$

where $Y(g^j u, z_0)$ is an operator on V;

$$[L(m), L(n)] = (m-n)L(m+n) + \frac{1}{12}(m^3 - m)\delta_{m+n,0}(\operatorname{rank} V) \tag{7.20}$$

for $m, n \in \mathbb{Z}$, where

$$L(n) = \omega_{n+1} \quad \text{for } n \in \mathbb{Z}, \text{ i.e., } Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}; \tag{7.21}$$

$$L(0)w = nw = (\operatorname{wt} w)w \quad \text{for } w \in M_n \ (n \in \mathbb{Q}); \tag{7.22}$$

$$\frac{\mathrm{d}}{\mathrm{d}z} Y(v,z) = Y(L(-1)v, z). \tag{7.23}$$

This completes the definition. We denote this module by (M, Y) (or briefly, by M).

We now work in the setting of Remarks 2.2, 4.2 and 4.5. In particular, L is a nondegenerate even lattice; ν is an isometry of L such that $\nu^p = 1$; $\boldsymbol{h} = L \otimes_{\mathbb{Z}} \mathbb{C}$; $\boldsymbol{h}_* = 0$; $M(1)$ and $S[\nu]$ are canonical irreducible modules for $\hat{\boldsymbol{h}}$ for $\hat{\boldsymbol{h}}[\nu]$ respectively; $\hat{L}$ and $\hat{L}_\nu$ are two central extensions of L by the finite cyclic group $\langle \kappa \mid \kappa^q = 1 \rangle$, with commutator maps c_0 and c_0^ν, respectively, given by (2.9) and (2.10); $c(\alpha, \beta) = (-1)^{\langle \alpha, \beta \rangle}$ and $c_\nu(\alpha, \beta) = \prod_{s=0}^{p-1}(-\omega_p^s)^{\langle \nu^s \alpha, \beta \rangle}$ for $\alpha, \beta \in L$; $\hat{\nu}$ is a automorphism of both $\hat{L}$ and $\hat{L}_\nu$ lifting ν such that $\hat{\nu} a = a$ if $\nu \bar{a} = \bar{a}$ for $a \in \hat{L}$ or $\hat{L}_\nu$ and such that $\hat{\nu}^p = 1$; $V_L = M(1) \otimes \mathbb{C}\{L\}$; $\mathbf{1} = \iota(1)$; $\omega = \frac{1}{2}\sum_{r=1}^{d} \beta_r(-1)^2$, where $\{\beta_1, \ldots, \beta_d\}$ is an orthonormal basis of $\boldsymbol{h}$; $Y(\cdot, z) = Y_*(\cdot, z)$ is the linear map in (3.41) (see also Remark 3.2):

$$\begin{aligned} &V_L \to (\operatorname{End} V_L)[[z, z^{-1}]] \\ &v \mapsto Y(v,z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1} \quad (v_n \in \operatorname{End} V_L); \end{aligned} \tag{7.24}$$

T is a $\hat{L}_\nu$-module as given in Remark 4.2; $V_L^T = S[\nu] \otimes T$ (see (4.25)); $Y_\nu(\cdot, z) = Y_*^\nu(\cdot, z)$ is the linear map in (4.46) (cf. Remark 4.5):

$$\begin{aligned} &V_L \to (\operatorname{End} V_L^T)[[z^{1/p}, z^{1/p}]] \\ &v \mapsto Y_\nu(v,z) = \sum_{n \in (1/p)\mathbb{Z}} v_n^\nu z^{-n-1} \quad (v_n^\nu \in \operatorname{End} V_L^T). \end{aligned} \tag{7.25}$$

It is proved in [14] (see also [2]) that $(V_L, Y, \mathbf{1}, \omega)$ is a vertex algebra of rank equal to $l = \operatorname{rank} L$. From (3.10) and (3.15) we see that $\hat{\nu}$ fixes $\mathbf{1}$ and ω:

$$\hat{\nu}\mathbf{1} = \mathbf{1}, \qquad \hat{\nu}\omega = \omega. \tag{7.26}$$

Also,

$$\hat{\nu} Y(v,z) \hat{\nu}^{-1} = Y(\hat{\nu} v, z), \quad v \in V_L, \tag{7.27}$$

from (3.48), that is, $\hat{\nu}$ is an automorphism of the vertex algebra V_L, and it has period p.

Theorem 7.1 ([19, 13, 20]). *Let L be an even lattice as in Remark 2.2. and let T be an $\hat{L}_\nu$-module as in Remark 4.2. Then the space (V_L^T, Y_ν) is a $\hat{\nu}$-twisted V_L-module. Moreover, V_L^T is irreducible if and only if T is an irreducible $\hat{L}_\nu$-module.*

Proof. Recall (4.59) with $\psi = 1$, Corollary 5.5 and Propositions 6.1 and 6.4 with $\boldsymbol{h}_* = 0$. We need only prove (7.16). Let $u = u^* \otimes \iota(a)$ for $u^* \in M(1)$ and $a \in \hat{L}$. Let

$$v = \sum_{r=0}^{p-1} \omega_p^{-rj} \hat{\nu}^r u \in V_L^j.$$

Then (7.16) follows from the fact that $Y_\nu(u, z) \in (\operatorname{End} V_L^T)[[z^{1/p}, z^{-1/p}]]$ (see Remark 4.5) and (4.52). □

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